Numerical homogenisation of microstructured continua and plates

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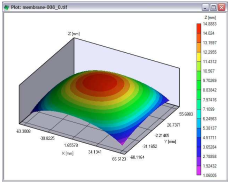
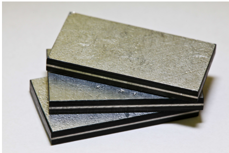
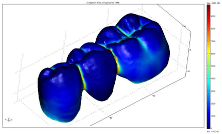
Outline



① Motivation

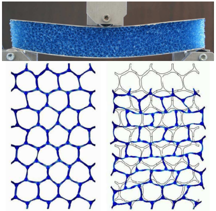


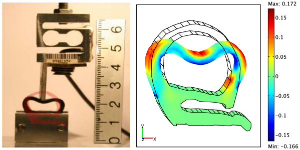


1 Motivation

Material modelling

- Basis for simulations of components and structures
- Complementary tool to experiments
- Understanding and prediction of the mechanical behaviour
- Optimisation of the material's structure

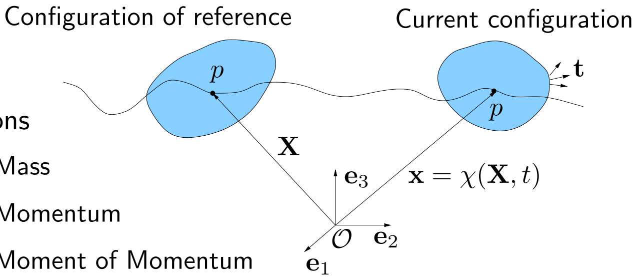




Theoretical basics: modelling of materials

- Continuum mechanics

  Description of the motion and deformation of a material body exposed to forces and torques



- Kinematics
- Balance equations
  - Balance of Mass
  - Balance of Momentum
  - Balance of Moment of Momentum
  - Balance of Energy (1^{st} law of thermodynamics)
  - Balance of Entropy (2^{nd} law of thermodynamics)
- Material law, e. g. stress as function of strain

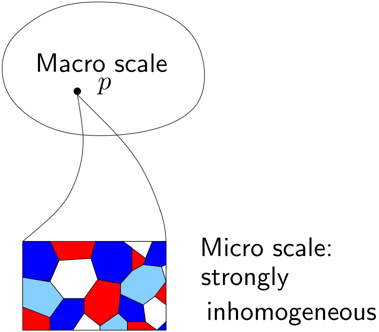
① Motivation

Continuum mechanics

- Macroscopic models

- Microstructure / atomistic structure not included

- Effective properties (can be derived from microscale models by homogenisation)



① Motivation

Smallest entity: material point

- Carrier of the physical properties
- Represented by a mathematical point
- Continuously distributed

$\Rightarrow$ Mathematical methods of differential calculus

$\Rightarrow$ Phenomenological approach

① Motivation

Example:
quasi-static mechanical behaviour of a structure

- Balance of momentum

$$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \mathbf{0}$$

- Material model

$$\boldsymbol{\sigma} = \mathbf{f}(\boldsymbol{\varepsilon}, \mathbf{q})$$

- $\mathbf{q}$: internal variables, e. g. plastic strain etc.

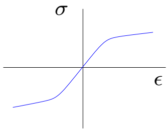
- The functional $\mathbf{f}$ is determined either by experiments or by microscale investigations (homogenisation)

Theoretical basics: modelling of materials

Rate independent

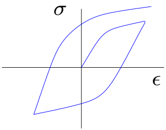
Without equilibrium hysteresis:

elastic



With equilibrium hysteresis:

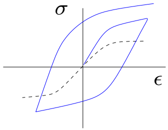
elasto-plastic



Rate dependent

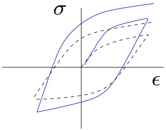
Without equilibrium hysteresis:

viscoelastic



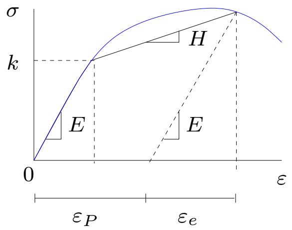
With equilibrium hysteresis:

elasto-viscoplastic



cf. P. Haupt, Continuum Mechanics and Theory of Materials, Springer-Verlag, 2000

Elastoplasticity: 1-dimensional tension test



- Decomposition of the strain:
  $\varepsilon = \varepsilon_{\mathrm{e}} + \varepsilon_{\mathrm{p}}$
- Yield function:
  $F = \sigma - k \leq 0$
  $F = 0$: plastic
  $F < 0$: elastic

- Elastic part: $\sigma = E\,(\varepsilon - \varepsilon_p) = E\,\varepsilon_e$
- Plastic part: $F = 0$ and loading

$$
\begin{aligned}
\dot{\varepsilon} &= \lambda \frac{\partial F(\sigma, k)}{\partial \sigma} \\
\dot{k} &= H\,\sigma\,\dot{\varepsilon_p}
\end{aligned}
$$

λ : plastic multiplier

$\Rightarrow$ consistency condition $\dot{F} = 0$ driving yielding

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Elastoplasticity

- Stress

$$\boldsymbol{\sigma} = K\ \mathrm{tr}(\boldsymbol{\varepsilon})\,\mathbf{I} + 2\,G\,(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p)^D$$

where $(\diamond)^D = (\diamond) - \dfrac{1}{3}\ \mathrm{tr}(\diamond)\,\mathbf{I}$

- Von Mises yield function:

$$F(\boldsymbol{\sigma}, \mathbf{X}) = \frac{1}{2}(\boldsymbol{\sigma} - \mathbf{X})^D : (\boldsymbol{\sigma} - \mathbf{X})^D - \frac{1}{3}\,k^2 \begin{cases} < 0 : \text{elastic} \\ = 0 : \text{plastic} \end{cases}$$

- Plastic deformations:

$$\dot{\boldsymbol{\varepsilon}}_p = \lambda\,\mathbf{N} = \lambda\,\frac{(\boldsymbol{\sigma} - \mathbf{X})^D}{||(\boldsymbol{\sigma} - \mathbf{X})^D||} \quad \text{for } F = 0 \text{ and } \quad \text{loading}$$

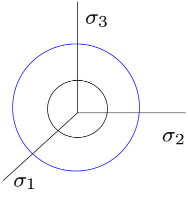
Elastoplasticity

- Hardening

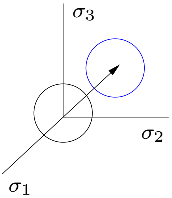
  * Isotropic hardening: $k = k(\boldsymbol{\varepsilon}_p)$
  * Kinematic hardening: $\mathbf{X}$ backstress tensor

$$\dot{\mathbf{X}} = c\,\dot{\boldsymbol{\varepsilon}}_p - b\,\dot{s}\,\mathbf{X} = \lambda\,(c\,\mathbf{N} - b\sqrt{2/3}\,\mathbf{X})$$

with s the accumulated plastic strain: $\dot{s} = \left(\frac{2}{3}\,\dot{\boldsymbol{\varepsilon}}_p : \dot{\boldsymbol{\varepsilon}}_p\right)^{1/2}$



Isotropic



Kinematic

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Viscoplasticity

- Stress: $$\boldsymbol{\sigma} = K \, \mathrm{tr}(\boldsymbol{\varepsilon}) \, \mathbf{I} + 2 \, G \, (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p)^D$$

  where $(\diamond)^D = (\diamond) - \dfrac{1}{3} \, \mathrm{tr}(\diamond) \, \mathbf{I}$

- Von Mises yield function:

$$F(\boldsymbol{\sigma}, \mathbf{X}) = \frac{1}{2}(\boldsymbol{\sigma} - \mathbf{X})^D : (\boldsymbol{\sigma} - \mathbf{X})^D - \frac{1}{3} \, k^2 \begin{cases} < 0 : \text{elastic} \\ \\ \geq 0 : \text{visco-pl.} \end{cases}$$

- Plastic deformations:

$$\dot{\boldsymbol{\varepsilon}}_p = \lambda \, \mathbf{N} = \lambda \, \frac{(\boldsymbol{\sigma} - \mathbf{X})^D}{||(\boldsymbol{\sigma} - \mathbf{X})^D||} \quad \text{for } F = 0 \text{ and } \quad \text{loading}$$

- Hardening model, cf. elastoplasticity

$$k = k(\boldsymbol{\varepsilon}_p),$$
$$\mathbf{X} = \mathbf{X}(\boldsymbol{\varepsilon}_p)$$

Numerical procedure

Solve balance of momentum and constitutive model for displacements

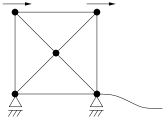
$$\boxed{\begin{array}{c} \operatorname{div} \boldsymbol{\sigma} = 0 \\ \boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \mathbf{q}) \end{array}}$$

$\Downarrow$ Principle of the virtual work,
Galerkin Procedure

$$\boxed{\begin{array}{c} \int_\Omega \delta\boldsymbol{\varepsilon} : \boldsymbol{\sigma} \, \mathrm{d}v = \int_{\Gamma_t} \delta\mathbf{u} \cdot \mathbf{t} \, \mathrm{d}a \\ \boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \mathbf{q}) \end{array}}$$

$\Downarrow$

FEM: Discretisation and choice of ansatz for displacement $\mathbf{u}$



$$\mathbf{u}(\mathbf{x}) = \sum_i N_i(\mathbf{x}) \, \mathbf{u}_i$$

with N_i: shape functions
$\mathbf{u}_i$: nodal values of $\mathbf{u}_i$

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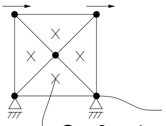
Numerical procedure

- Equilibrium condition after discretisation

$$\int \operatorname{grad}(\sum N_i \delta \mathbf{u}_i) : \boldsymbol{\sigma}\left(\operatorname{grad}(\sum N_j \mathbf{u}_j), \mathbf{q}\right) \, \mathrm{d}v = \int \sum N_i \, \delta \mathbf{u}_i \cdot \mathbf{t} \, \mathrm{d}a$$

- Integration by quadrature



$$\int f(x) \, \mathrm{d}v = \sum_g w_g \, f(\mathbf{x}_g)$$

w_g: Integration weights

x_g: Integration points / Gauß points

- Solution of the constitutive model at each Gauß point

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}\left(\underbrace{\operatorname{grad}\left(\sum N_j(\mathbf{x}_G) \, \mathbf{u}_j\right)}_{\text{strains at } \mathbf{x}_G}, \; \underbrace{\mathbf{q}(\mathbf{x}_G)}_{\text{int. variable}} \right)$$

and evolution equations

Numerical procedure

- Computation of $\boldsymbol{\sigma}$ at $\mathbf{x}_G$ needs the solution of the elastoplastic/ elastic-viscoplastic constitutive model

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}\left(\underbrace{\operatorname{grad}\left(\sum N_j(\mathbf{x}_G)\,\mathbf{u}_j\right)}_{\text{strains at } \mathbf{x}_G},\ \underbrace{\boldsymbol{\epsilon}_p(\mathbf{x}_G)}_{\text{int. variable}} \right)$$

- Time integration of the evolution equations required

  (e. g. implicite Euler scheme)

$$\begin{aligned} \frac{\boldsymbol{\epsilon}_p - \boldsymbol{\epsilon}_p^{old}}{\Delta t} &= \lambda \frac{\partial F(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} \\ F(\boldsymbol{\sigma}) &= 0 \end{aligned}$$

$\Rightarrow$ Displacement $\mathbf{u}_j$ and int. variables $\boldsymbol{\epsilon}_p$ at new time step

Multi Level Newton Algorithm: Principle

- Coupled equations usually solved by *radial return algorithm*
- Here: Multi Level Newton Algorithm (MLNA)
  cf. Hartmann et al. (1997), Ellsiepen & Hartmann (2001)

- Weak formulation: discretisation $\mathbf{u} = \sum \mathbf{N}_j \, \mathbf{u}_j$; $\delta\mathbf{u} = \sum \mathbf{N}_j \, \delta\mathbf{u}_j$
  - Global level: global equilibrium condition

$$\int_V \operatorname{grad} \delta\mathbf{u} : \boldsymbol{\sigma} \, \mathrm{d}V = \int_V \delta\mathbf{u} \cdot \mathbf{f} \, \mathrm{d}V = 0 \quad \Leftrightarrow \quad \mathbf{G}(\mathbf{u}_j, \, \mathbf{q}_G) = 0$$

  - Local level: evolution equations at Gauß points $\mathbf{x}_G$

$$\mathbf{A} \, \dot{\mathbf{q}}_G - \mathbf{r}(\mathbf{u}_j, \mathbf{q}_G) = 0 \qquad \Leftrightarrow \qquad \mathbf{l}(\mathbf{u}_j, \mathbf{q}_G) = 0$$

Multilevel Newton Algorithm (MLNA)

- Solve the coupled nonlinear equations

$$\mathbf{G}(\mathbf{u},\mathbf{q}) \,=\, 0, \qquad \mathbf{l}(\mathbf{u},\mathbf{q}) \,=\, 0$$

- Newton iteration

$$\left[\begin{array}{c|c} \frac{\partial \mathbf{G}}{\partial \mathbf{u}} & \frac{\partial \mathbf{G}}{\partial \mathbf{q}} \\ \hline \frac{\partial \mathbf{l}}{\partial \mathbf{u}} & \frac{\partial \mathbf{l}}{\partial \mathbf{q}} \end{array}\right] \cdot \left[\begin{array}{c} \mathrm{d}\mathbf{u} \\ \hline \mathrm{d}\mathbf{q} \end{array}\right] = - \left[\begin{array}{c} \mathbf{R}_{\mathbf{u}} \\ \hline \mathbf{R}_{\mathbf{q}} \end{array}\right]$$

$\partial\mathbf{G}/\partial\mathbf{u}$: sparse fem matrix (global)

$\partial\mathbf{l}/\partial\mathbf{q}$: block diagonal matrix (local)

$\partial\mathbf{G}/\partial\mathbf{q}$, $\partial\mathbf{l}/\partial\mathbf{u}$: coupling

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Multilevel Newton Algorithm (MLNA)

- Step 1: local level
- Solve evolution equations for a given displacement, i. e. $\mathrm{d}\mathbf{u} = \mathbf{0}$
- Second line becomes

$$\frac{\partial \mathbf{l}}{\partial \mathbf{q}} \cdot \mathrm{d}\mathbf{q} = -\mathbf{R}_{\mathbf{q}} \quad \Rightarrow \mathbf{R}_{\mathbf{q}} = \mathbf{0}$$

$$\mathrm{d}\mathbf{q} = -\left(\frac{\partial \mathbf{l}}{\partial \mathbf{q}}\right)^{-1} \cdot \mathbf{R}_{\mathbf{q}}$$

- Update of the internal variables

$$\mathbf{q} += \mathrm{d}\mathbf{q}$$

Multilevel Newton Algorithm (MLNA)

- Step 2: $\mathbf{R_q} = \mathbf{0}$ after update of $\mathbf{q}$
- Second line is re-written as

$$\frac{\partial \mathbf{l}}{\partial \mathbf{u}} \cdot \mathrm{d}\mathbf{u} + \frac{\partial \mathbf{l}}{\partial \mathbf{q}} \cdot \mathrm{d}\mathbf{q} = \mathbf{0}$$

$$\mathrm{d}\mathbf{q} = \left(\frac{\partial \mathbf{l}}{\partial \mathbf{q}} \right)^{-1} \cdot \left(-\frac{\partial \mathbf{l}}{\partial \mathbf{u}} \right) \cdot \mathrm{d}\mathbf{u}$$

- Interpretation:
  internal variables are linked to the displacements

$$\Rightarrow \quad \mathrm{d}\mathbf{q} = \mathrm{d}\mathbf{q}(\mathbf{u})$$

cf. Hartmann et al. (1997), Ellsiepen & Hartmann (2001)

Multilevel Newton Algorithm (MLNA)

- Step 3: global level
- Solve equilibrium condition (first line)

$$\frac{\partial \mathbf{G}}{\partial \mathbf{u}} \cdot \mathrm{d}\mathbf{u} + \frac{\partial \mathbf{G}}{\partial \mathbf{q}} \cdot \mathrm{d}\mathbf{q} = -\mathbf{R}_{\mathbf{u}}$$

$$\left(\frac{\partial \mathbf{G}}{\partial \mathbf{u}} + \frac{\partial \mathbf{G}}{\partial \mathbf{q}} \cdot \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}\mathbf{u}} \right) \cdot \mathrm{d}\mathbf{u} = -\mathbf{R}_{\mathbf{u}}$$

- Consistent tangent

$$\Rightarrow \mathbf{C} = \frac{\partial \mathbf{G}}{\partial \mathbf{u}} + \frac{\partial \mathbf{G}}{\partial \mathbf{q}} \cdot \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}\mathbf{u}}$$

with $\mathrm{d}\mathbf{q}/\mathrm{d}\mathbf{u}$ from step 2

Multilevel Newton Algorithm (MLNA)

- Step 4: equilibrium
- Sove globe FEM system and update $\mathbf{u}$

$$\mathrm{d}\mathbf{u} = -\mathbf{C}^{-1}\,\mathbf{R}_{\mathbf{u}}, \quad \mathbf{u} += \mathrm{d}\mathbf{u}$$

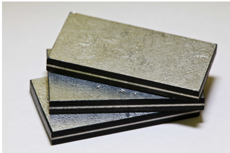
- Goto Step 1 until $||\mathbf{R}_{\mathbf{u}} < TOL_1||$ and $||\mathbf{R}_{\mathbf{q}} < TOL_2||$
- Next load step

③ Homogenisation methods

Modelling of materials

Here: hybrid laminates

- Arbitrary stacking of metal and polymer layers
  - Metal: Titanium or Aluminium
  - Polymer: reinforced by endless fibers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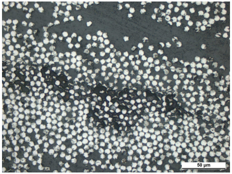

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(1): DLR, Köln, M. Bartsch, J. Hausmann, K. Schulze;
(2): TU Chemnitz, L. Kroll, S. Nendel

Modelling of materials

- Complex microstructure:
  use of micromechanics and homogenisation methods
  - ∗ Averaging methods (Mori-Tanaka, self-consistent methods ...)
  - ∗ Bounds (Voigt-Reuss, Hashin-Strikman ...)
- Concept of scale separation, cf. Neumat-Nasser & Hori (1999)

$$\lambda_{macro} \gg \lambda_{micro}$$

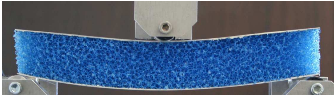
  - ∗ Macroscopic continuum considered as homogeneous
  - ∗ Heterogeneities considered in the microscale

Homogenisation methods

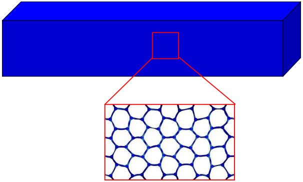
- Definition of a Representative Volume Element (RVE)
  Hill (1963), Hashin (1964, 1983), Kröner (1977), Willis (1981),
  Nemat-Nasser (1986)

  * The RVE is "statistically representative of the local continuum properties", Nemat-Nasser & Hori (1999)

$$\lambda_{macro} \gg \lambda_{RVE} \gg \lambda_{micro}$$



$\Longrightarrow$



- Hill-Mandel condition: macro stress power = micro stress power

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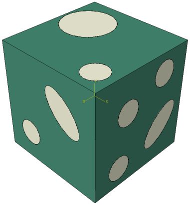
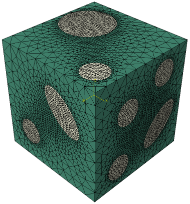
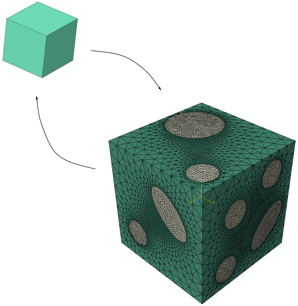
Homogenisation methods

- Homogenisation methods







- Analytical homogenisation, cf. Ponte Castañeda (2002), Suquet (1985)
- Numerical homogenisation, cf. Zohdi & Wriggers (2001), Segurado et al. (2012) ...
- Numerical concurrent homogenisation (FE^2), cf. Feyel & Chaboche (2001), Forest (1998), Kouznetsova (2002), Ebinger (2009), Jänicke (2010)...

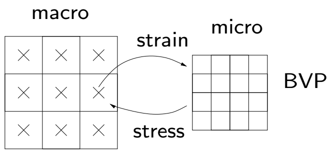
Homogenisation methods

- Macroscale computation / weak format of equilibrium condition

$$\int \delta\boldsymbol{\varepsilon} : \boldsymbol{\sigma}\,\mathrm{d}v = \int \delta\mathbf{u}\cdot\mathbf{t}\,\mathrm{d}a$$

- Evaluation of $\boldsymbol{\sigma}$ by homogenisation
  - $\Rightarrow$ at each Gauss point
    - ∗ Projection of the macro strain to a RVE
    - ∗ Microscale FEM computation of inhomogeneous BVP on RVE
    - ∗ Homogenisation of local stresses to macro stress

$$\bar{\boldsymbol{\sigma}} = \frac{1}{V}\int_V \boldsymbol{\sigma}\mathrm{d}v$$

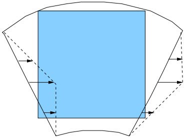


Projection of the deformations in the RVE

- Taylor- or Voigt-assumption: constant deformation
- Sachs- or Reuss-assumption: constant stress

Limitations: Taylor-assumption: overestimation of the results
Sachs-assumption: underestimation of the results

- Projection on the boundary of the RVE, fluctuations
$\Delta \mathbf{u} = \mathrm{Grad}\, \bar{\mathbf{u}} \cdot \Delta \mathbf{X} + \Delta \widetilde{\mathbf{u}}$



Periodic boundary conditions
$\tilde{\mathbf{x}}^{+} - \tilde{\mathbf{x}}^{-} = \bar{\mathbf{F}} \cdot (\tilde{\mathbf{X}}^{+} - \tilde{\mathbf{X}}^{-})$

- Equivalence between the macroscopic and the mesoscopic deformation gradient

$$\bar{\mathbf{F}} = \frac{1}{\mathcal{B}_0} \int_{\mathcal{B}_0} \mathbf{F}_m \, \mathrm{d}\mathcal{B}_0$$

cf. Kouznetsova (2004), Coenen et al. (2010)

Applications of FE^2

- Complex microstructures, strong coupling between scales
  cf. presentation by D. Balzani

- Extended continua
  e. g. gradient continua, Cosserat continua, micromorphic continua

  * Projection of higher order deformation measures, e. g. curvature
  * Replace phenomenological model for hyper-stresses etc.
  * Simple microstructures yield governing effects

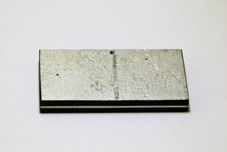
- cf. Forest et al., Geers et al., Jänicke et al.

③ Numerical Homogenisation for plates

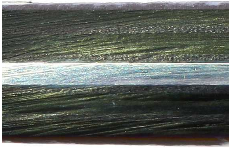
- Plates
  Altenbach et al. (1996), Reddy (1997), Bischoff (1999)
- Numerical homogenisation
  Feyel & Chaboche (2000), Forest & Trinh (2011)
- Homogenisation for plates
  Cecci & Sab (2002), Geers & al. (2007),
  Landervik & Larsson (2008), Coenen & al. (2010),
  Helfen & Diebels (2011), Lebee & Sab (2012)

③ Scale separation

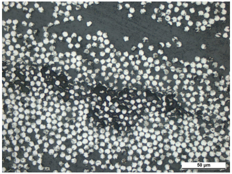
- Macroscale



- Mesoscale



- Microscale



Samples produced by DLR, Köln, M. Bartsch; TU Chemnitz, L. Kroll

Principle of the scale separation cf. Nemat-Nasser & Hori (1999)

$$\lambda_{macro} \gg \lambda_{meso} \gg \lambda_{micro}$$

$\Rightarrow$ Numerical homogenisation only in both longitudinal directions

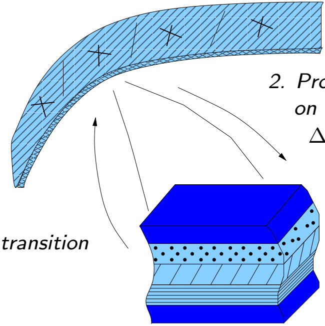
$\Rightarrow$ Full discretization of the thickness direction / layers stacking order

② Principle of a FE^2 method

1. Plate kinematics

$\bar{\mathbf{u}} = f(\bar{u}_0, \bar{v}_0 \ldots)$

Macroscale



Plate

2. Projection of the deformations on the boundary of the RVE

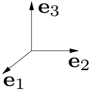
$\Delta\mathbf{u} = \mathrm{Grad}\,\bar{\mathbf{u}} \cdot \Delta\mathbf{X} + \Delta\tilde{\mathbf{u}}$

Mesoscale

3-D RVE

4. Meso-macro-transition

3. Resolution of the BVP (Boundary Value Problem)

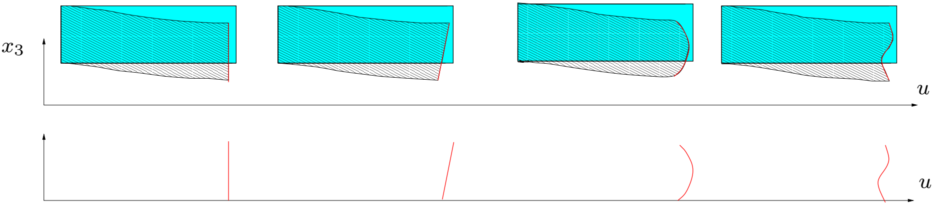


Equilibrium equation	Constitutive equation
$\mathrm{div}\,\mathbf{P} = 0$	$\mathbf{P} = \mathbf{f}(\mathbf{F})$

$\bar{\diamond}$: macroscale, $\diamond$: mesoscale; cf. Chambon & Diebels, in *PAMM*, 2011

2.1 Different plate theories

- Polynomial approximation of the deformation of the cross section



0	1	2	3
Love-Kirchhoff	Reissner-Mindlin	2 order	...

- Which approximation is required?
- What is the resulting projection?

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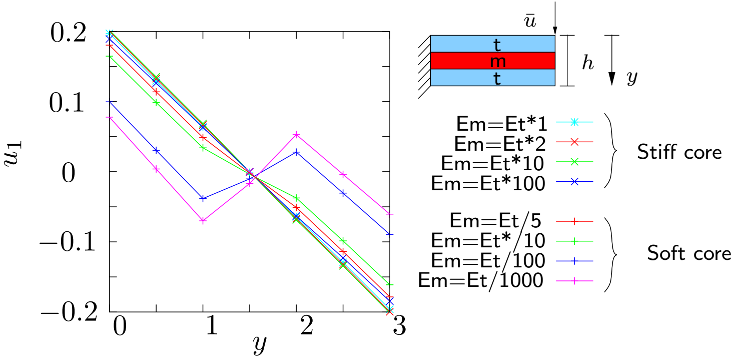
2.1 Fully resolved problem vs. polynoms

- 3 layers, bending

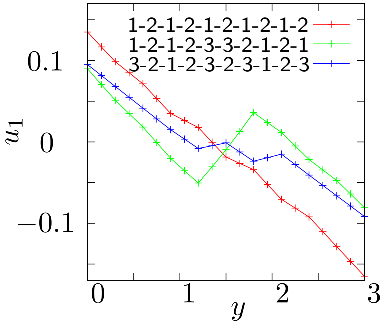


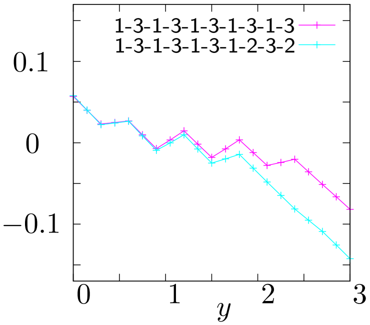
⇒ Displacement u_1: 3^{rd} oder polynomial function for soft cores

⇒ Displacement u_1: linear function for stiff cores

2.1 Fully resolved problem vs. polynoms

- 10 layers ($E_1 > E_2 > E_3$)





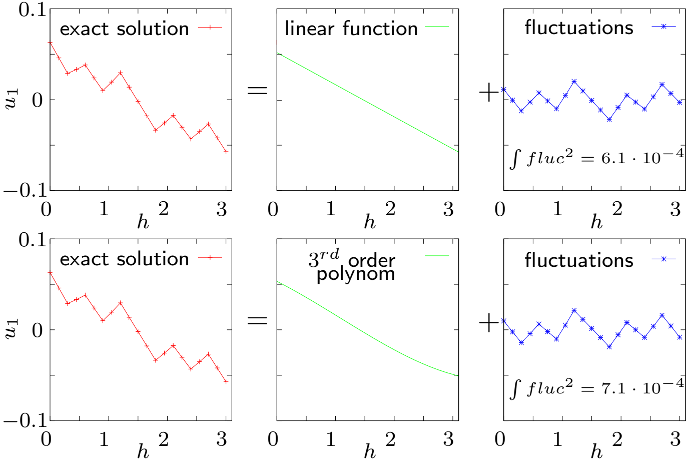
$\Rightarrow$ Displacement u_1 approximated by odd polynomials

$\Rightarrow$ The bigger the number of layers, the better the approximation with a linear function

3.1 Fully resolved problem vs. polynoms

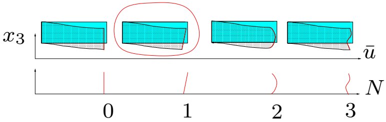
Approximation by

- Linear function and fluctuation field
- Cubic polynom and fluctuation field



Mindlin plate theory

5 degrees of freedom $(\bar{u}_0, \bar{v}_0, \bar{w}_0, \bar{\varphi}_1, \bar{\varphi}_2)$
no thickness change (cf. Altenbach (1996))



- Displacement field

$$
\begin{aligned}
\bar{u}(\bar{x}_1, \bar{x}_2, \bar{x}_3) &= \bar{u}_0(\bar{x}_1, \bar{x}_2) + x_3\, \bar{\varphi}_1(\bar{x}_1, \bar{x}_2) \\
\bar{v}(\bar{x}_1, \bar{x}_2, \bar{x}_3) &= \bar{v}_0(\bar{x}_1, \bar{x}_2) + x_3\, \bar{\varphi}_2(\bar{x}_1, \bar{x}_2) \\
\bar{w}(\bar{x}_1, \bar{x}_2, \bar{x}_3) &= \bar{w}_0(\bar{x}_1, \bar{x}_2)
\end{aligned}
$$

- Deformations

$$
\bar{\varepsilon}_{11} = \frac{\partial \bar{u}_0}{\partial \bar{x}_1} + x_3 \frac{\partial \bar{\varphi}_1}{\partial \bar{x}_1} \qquad \bar{\gamma}_{12} = \frac{\partial \bar{u}_0}{\partial \bar{x}_2} + \frac{\partial \bar{v}_0}{\partial \bar{x}_1} + x_3 \left(\frac{\partial \bar{\varphi}_1}{\partial \bar{x}_2} + \frac{\partial \bar{\varphi}_2}{\partial \bar{x}_1} \right)
$$

$$
\bar{\varepsilon}_{22} = \frac{\partial \bar{v}_0}{\partial \bar{x}_2} + x_3 \frac{\partial \bar{\varphi}_2}{\partial \bar{x}_2} \qquad \bar{\gamma}_{13} = \frac{\partial \bar{w}_0}{\partial \bar{x}_1} + \bar{\varphi}_1 \qquad \bar{\gamma}_{23} = \frac{\partial \bar{w}_0}{\partial \bar{x}_2} + \bar{\varphi}_2
$$

Voigt notation

$$
\bar{\boldsymbol{\varepsilon}}_0 = (\bar{\varepsilon}_{11}, \bar{\varepsilon}_{22}, \bar{\varepsilon}_{12})^T; \quad \bar{\boldsymbol{\kappa}} = (\bar{\kappa}_{11}, \bar{\kappa}_{22}, \bar{\kappa}_{12})^T; \quad \bar{\boldsymbol{\gamma}} = (\bar{\gamma}_{13}, \bar{\gamma}_{23})^T
$$

Mindlin plate theory

5 degrees of freedom $(\bar{u}_0, \bar{v}_0, \bar{w}_0, \bar{\varphi}_1, \bar{\varphi}_2)$

- Generalised forces (stress resultants)

$$\bar{\mathcal{N}} = \Big(\underbrace{\bar{N}_{11}, \bar{N}_{22}, \bar{N}_{12},}_{\text{stress resultants}} \quad \underbrace{\bar{M}_{11}, \bar{M}_{22}, \bar{M}_{12},}_{\text{moment resultants}} \quad \underbrace{\bar{Q}_1, \bar{Q}_2}_{\text{shear forces}} \Big)^T$$

- Relation between stresses and forces, moments and shear forces

$$\bar{\mathcal{N}} = [\bar{N}_{ij}, \bar{M}_{ij}, \bar{Q}_i]^T = \int_{-h/2}^{h/2} [P_{ij}, P_{ij}\, x_3, P_{3i}]^T \,\mathrm{d}x_3, \quad i,j = 1,2$$

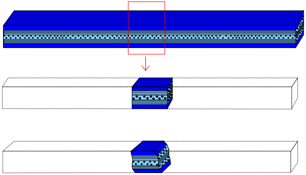
- Constitutive equations

$$\bar{\mathcal{N}} = \mathbb{C} \cdot \begin{bmatrix} \bar{\boldsymbol{\varepsilon}}_0 \\ \bar{\boldsymbol{\kappa}} \\ \bar{\boldsymbol{\gamma}} \end{bmatrix} \Leftrightarrow \begin{bmatrix} \bar{\mathbf{N}} \\ \bar{\mathbf{M}} \\ \bar{\mathbf{Q}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbb{C}}^1 & \bar{\mathbb{C}}^2 & \mathbf{0} \\ \bar{\mathbb{C}}^2 & \bar{\mathbb{C}}^3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbb{C}}^4 \end{bmatrix} \cdot \begin{bmatrix} \bar{\boldsymbol{\varepsilon}}_0 \\ \bar{\boldsymbol{\kappa}} \\ \bar{\boldsymbol{\gamma}} \end{bmatrix}$$

Representative Volume Element (RVE)

Numerical homogenisation:
Longitudinal direction

Thickness direction: discretisation
of the layer stacking order/ full resolution



On the RVE scale

- Balance equation between the internal and external work in RVE

$$\int_{\mathcal{B}} \mathbf{P} : \operatorname{grad} \delta\mathbf{u} \, \mathrm{d}v = \int_{\mathcal{B}} \rho \, \delta\mathbf{u} \cdot \mathbf{b} \, \mathrm{d}v + \int_{\partial\mathcal{B}} \delta\mathbf{u} \cdot \mathbf{t} \, \mathrm{d}a$$

- Constitutive equation on the mesoscale for the different materials

$$\mathbf{P} = \mathcal{F}(\mathbf{F})$$

where $\mathbf{P}$ is the mesoscopic 1. Piola-Kirchhoff stress tensor

Meso-Macro-Transition

- Hill-Mandel condition, cf. Feyel & Chaboche (2000), Forest (2002)

$$\frac{1}{\mathcal{B}_0}\int_{\mathcal{B}_0}\mathbf{P}:\delta\mathbf{F}\,\mathrm{d}\mathcal{B}_0=\bar{\mathbf{P}}:\delta\bar{\mathbf{F}}$$

- For plates, cf. Landervik & Larsson (2008), Coenen et al. (2010)

$$\frac{1}{\mathcal{S}_0}\int_{\mathcal{B}_0}\mathbf{P}:\delta\mathbf{F}\,\mathrm{d}\mathcal{B}_0=\bar{\mathbf{N}}:\delta\bar{\boldsymbol{\varepsilon}}_0+\bar{\mathbf{M}}:\delta\bar{\boldsymbol{\kappa}}+\bar{\mathbf{Q}}:\delta\bar{\boldsymbol{\gamma}}$$

with

$$\bar{\boldsymbol{\varepsilon}}_0=(\bar{\varepsilon}_{11},\bar{\varepsilon}_{22},\bar{\varepsilon}_{12})^T;\;\;\bar{\boldsymbol{\kappa}}=(\bar{\kappa}_{11},\bar{\kappa}_{22},\bar{\kappa}_{12})^T;\;\;\bar{\boldsymbol{\gamma}}=(\bar{\gamma}_{13},\bar{\gamma}_{23})^T$$

leads to

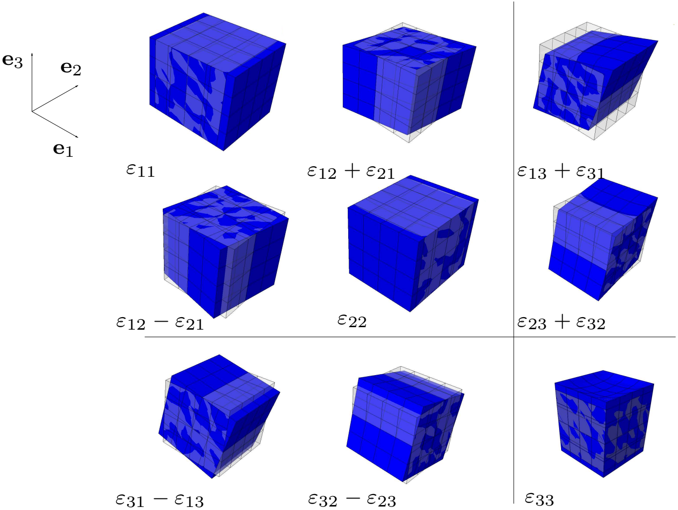
$$\bar{\mathcal{N}}=[\bar{N}_{ij},\bar{M}_{ij},\bar{Q}_i]^T=\int_{-h/2}^{h/2}[P_{ij},P_{ij}\,x_3,P_{3i}]^T\,\mathrm{d}x_3,$$

$\mathbf{P}$: Mesoscopic 1. Piola-Kirchhoff stress tensor

$\bar{\mathbf{N}}$, $\bar{\mathbf{M}}$ and $\bar{\mathbf{Q}}$: stress resultants

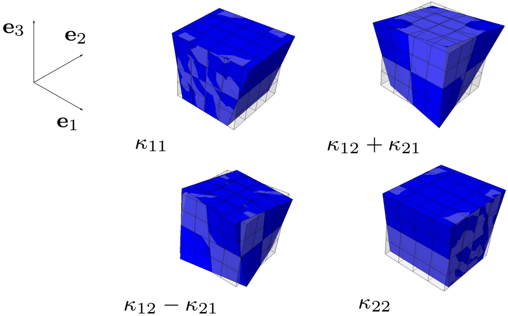
Projection modes

- 0. order deformation modes



Projection modes

- 1. order deformation modes



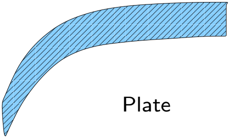
③ Multi-Level Newton Algorithm

cf. Rabat et al. (1979), Ellsiepen & Hartmann (2001),
Hartmann (2005), Hartmann et al. (2008),
Helfen & Diebels (2011)

③ Multi-Level Newton Algorithm (MLNA)

- Macro level: Plate theory

$$\begin{bmatrix} \bar{N}_{11,1} + \bar{N}_{12,2} \\ \bar{N}_{12,1} + \bar{N}_{22,2} \\ \bar{Q}_{1,1} + \bar{Q}_{2,2} + \mathfrak{N}(w_0) + q \\ \bar{M}_{11,1} + \bar{M}_{12,2} - \bar{Q}_1 \\ \bar{M}_{12,1} + \bar{M}_{22,2} - \bar{Q}_2 \end{bmatrix} = \mathbf{0}$$



- Weak formulation and fem discretisation

  * Global equilibrium condition $\mathbf{G} = \mathbf{0}$
    with $\bar{\mathcal{N}}_i$ defined at each macroscopic integration point

  * $\bar{\mathcal{N}}_i$ obtained by numerical homogenisation
    (replaces the constitutive equation)

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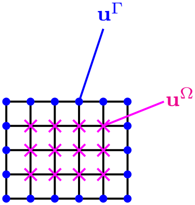
③ Multi-Level Newton Algorithm (MLNA)

- Local level, full resolution model of RVE (mesoscale, 3-D)
  - Dirichlet boundary conditions obtained by projection $\mathbf{u}^\Gamma$

$$\Delta\mathbf{u} = \operatorname{Grad}\bar{\mathbf{u}} \cdot \Delta\mathbf{X} + \Delta\widetilde{\mathbf{u}}$$

  - FEM model inside the RVE

$$-\int_{\mathcal{B}} \mathbf{P} : \operatorname{grad}\delta\mathbf{u}\,\mathrm{d}v + \int_{\mathcal{B}} \rho\,\delta\mathbf{u}\cdot\mathbf{b}\,\mathrm{d}v + \int_{\partial\mathcal{B}} \delta\mathbf{u}\cdot\mathbf{t}\,\mathrm{d}a = \mathbf{0}$$



- Dirichlet BVP due to the projection of strains

$$\Rightarrow \mathbf{l} = \mathbf{0}$$

③ Multi-Level Newton Algorithm (MLNA)

- Non-linear coupled equations

$$\mathbf{G}(\bar{\mathbf{u}}, \mathbf{u}) = 0, \qquad \mathbf{l}(\bar{\mathbf{u}}, \mathbf{u}) = 0$$

- Newton iteration, cf. Hartmann et al. (2008)

$$\left[\begin{array}{c|c} \dfrac{\partial \mathbf{G}}{\partial \bar{\mathbf{u}}} & \dfrac{\partial \mathbf{G}}{\partial \mathbf{u}} \\ \hline \dfrac{\partial \mathbf{l}}{\partial \bar{\mathbf{u}}} & \dfrac{\partial \mathbf{l}}{\partial \mathbf{u}} \end{array}\right] \cdot \left[\begin{array}{c} \mathrm{d}\bar{\mathbf{u}} \\ \hline \mathrm{d}\mathbf{u} \end{array}\right] = - \left[\begin{array}{c} \mathbf{R}_{\bar{\mathbf{u}}} \\ \hline \mathbf{R}_{\mathbf{u}} \end{array}\right]$$

$\mathbf{G}$: macro scale equilibrium; $\mathbf{l}$: meso scale equilibrium

$\partial \mathbf{G}/\partial \bar{\mathbf{u}}$: sparse fem matrix (global equilibrium);
$\partial \mathbf{l}/\partial \mathbf{u}$: block diagonal matrix (local equilibrium)
$\partial \mathbf{G}/\partial \mathbf{u}$, $\partial \mathbf{l}/\partial \bar{\mathbf{u}}$: coupling (projection / homogenisation)

③ Multi-Level Newton Algorithm (MLNA)

- Step 1: local equilibrium
- Solve local equilibrium for a given macro displacement $\mathrm{d}\bar{\mathbf{u}} = \mathbf{0}$

$$\frac{\partial \mathbf{l}}{\partial \mathbf{u}} \cdot \mathrm{d}\mathbf{u} = -\mathbf{R}_{\mathbf{u}}$$

$$\mathrm{d}\mathbf{u} = -\left(\frac{\partial \mathbf{l}}{\partial \mathbf{u}}\right)^{-1} \mathbf{R}_u$$

- Update

$$\mathbf{u} += \mathrm{d}\mathbf{u}$$

③ Multi-Level Newton Algorithm (MLNA)

- Step 2
  After update of $\mathbf{u}$ assume $\mathbf{R}_{\mathbf{u}} = 0$

$$
\begin{aligned}
\frac{\partial \mathbf{l}}{\partial \bar{\mathbf{u}}} \cdot \mathrm{d}\bar{\mathbf{u}} + \frac{\partial \mathbf{l}}{\partial \mathbf{u}} \cdot \mathrm{d}\mathbf{u} = & \quad \mathbf{0} \\
\mathrm{d}\mathbf{u} = \left(\frac{\partial \mathbf{l}}{\partial \mathbf{u}} \right)^{-1} \cdot \left(-\frac{\partial \mathbf{l}}{\partial \bar{\mathbf{u}}} \right) \cdot & \quad \mathrm{d}\bar{\mathbf{u}}
\end{aligned}
$$

- Relation between increments $\mathrm{d}\mathbf{u}$ and $\mathrm{d}\bar{\mathbf{u}}$

$$\Rightarrow \mathrm{d}\mathbf{u} = \mathrm{d}\mathbf{u}(\mathrm{d}\bar{\mathbf{u}})$$

③ Multi-Level Newton Algorithm (MLNA)

- Step 3: global equilibrium

$$\frac{\partial \mathbf{G}}{\partial \bar{\mathbf{u}}} \cdot \mathrm{d}\bar{\mathbf{u}} + \frac{\partial \mathbf{G}}{\partial \mathbf{u}} \cdot \mathrm{d}\mathbf{u} \;=\; \left(\frac{\partial \mathbf{G}}{\partial \bar{\mathbf{u}}} + \frac{\partial \mathbf{G}}{\partial \mathbf{u}} \cdot \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\bar{\mathbf{u}}} \right) \cdot \mathrm{d}\bar{\mathbf{u}} \;=\; -\mathbf{R}_{\bar{\mathbf{u}}}$$

- Required relations by the chain rule

$$\mathbf{G}(\bar{\mathbf{u}}, \mathbf{u}) \;=\; \mathbf{G}(\bar{\mathbf{u}}, \bar{\mathcal{N}}(\mathbf{P}(\mathbf{u})))$$

  * Equilibrium equations of the plate

$$\Rightarrow \frac{\partial \mathbf{G}}{\partial \bar{\mathcal{N}}}$$

  * Homogenisation

$$[\bar{N}_{ij}, \bar{M}_{ij}, \bar{Q}_i]^T = \int_{-h/2}^{h/2} [P_{ij}, P_{ij}\, x_3, P_{3i}]^T \,\mathrm{d}x_3 \quad \Rightarrow \quad \frac{\partial \bar{\mathcal{N}}}{\partial \mathbf{P}}$$

③ Multi-Level Newton Algorithm (MLNA)

* Equilibrium equation of the mesoscale (local part of the iteration)

$$\int_{\mathcal{B}} \mathbf{P} : \operatorname{grad} \delta\mathbf{u}\, \mathrm{d}v = \int_{\mathcal{B}} \rho\, \delta\mathbf{u} \cdot \mathbf{b}\, \mathrm{d}v + \int_{\partial\mathcal{B}} \delta\mathbf{u} \cdot \mathbf{t}\, \mathrm{d}a = \mathbf{0} \quad \Rightarrow \quad \frac{\partial \mathbf{P}}{\partial \mathbf{u}}$$

* Projection

$$\Delta\mathbf{u} = \operatorname{Grad} \bar{\mathbf{u}} \cdot \Delta\mathbf{X} + \Delta\widetilde{\mathbf{u}} \quad \Rightarrow \quad \frac{\partial \mathbf{u}}{\partial \bar{\mathbf{u}}}$$

- No direct relation between $\mathbf{G}$ and $\bar{\mathbf{u}}$, i. e. $\dfrac{\partial \mathbf{G}}{\partial \bar{\mathbf{u}}} = \mathbf{0}$
- Identification of the consistent tangent operator

$$\underbrace{\left(\frac{\partial \mathbf{G}}{\partial \mathbf{u}} \cdot \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\bar{\mathbf{u}}} \right)}_{\mathbf{K}} \cdot \mathrm{d}\bar{\mathbf{u}} = -\mathbf{R}_{\bar{\mathbf{u}}}$$

③ Multi-Level Newton Algorithm (MLNA)

- Step 4
  update $\bar{\mathbf{u}}$

$$\mathrm{d}\bar{\mathbf{u}} = -\mathbf{K}^{-1}\,\mathbf{R}_{\bar{\mathbf{u}}}$$

- goto 1 until $||\mathbf{R}_{\bar{\mathbf{u}}} < TOL_1||$ and $||\mathbf{R}_{\mathbf{u}} < TOL_2||$

$\longrightarrow$ next load step

④ Numerical results

Macroscale: plate $\Rightarrow$ non-commercial FE-software

Mesoscale/RVE: 3-D $\Rightarrow$ ABAQUS

Consideration of the thickness change

- Obvious contradiction in macro- and mesoscale assumptions

- Macroscale: 5 degrees of freedom

$\longrightarrow$ No thickness change of the plate

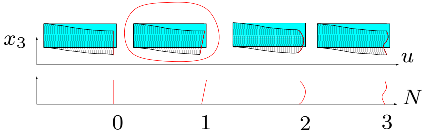
- Mesoscale: Plane stress

$\longrightarrow$ Thickness change of the RVE

$\Rightarrow$ Convergence problems in the general case!

Consideration of the thickness change

- Plate theory with thickness change



Displacement field

$$
\begin{aligned}
\bar{u}(\bar{x}_1, \bar{x}_2, \bar{x}_3) &= \bar{u}_0(\bar{x}_1, \bar{x}_2) + x_3\, \bar{\varphi}_1(\bar{x}_1, \bar{x}_2) \\
\bar{v}(\bar{x}_1, \bar{x}_2, \bar{x}_3) &= \bar{v}_0(\bar{x}_1, \bar{x}_2) + x_3\, \bar{\varphi}_2(\bar{x}_1, \bar{x}_2) \\
\bar{w}(\bar{x}_1, \bar{x}_2, \bar{x}_3) &= \bar{w}_0(\bar{x}_1, \bar{x}_2) + x_3\, \bar{\theta}_1\, (\bar{x}_1, \bar{x}_2) \;+\; x_3^2\, \bar{\theta}_2\, (\bar{x}_1, \bar{x}_2)
\end{aligned}
$$

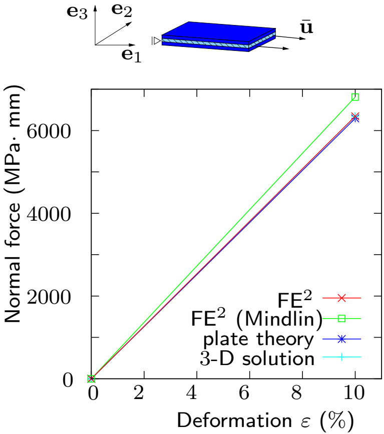
Deformations $\Rightarrow \bar{\varepsilon}_{33} \neq 0 = \bar{\theta}_1 \;+\; x_3\, 2\, \bar{\theta}_2$

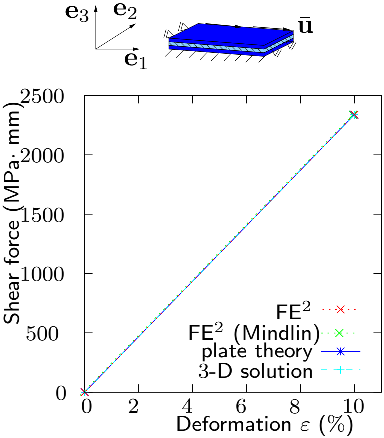
- Modification of the projection,
  thickness change as macroscopic internal variable

$$\Delta \mathbf{u} = \mathrm{Grad}\, \bar{\mathbf{u}} \cdot \Delta \mathbf{X} + \Delta \bar{u}_3 + \Delta \widetilde{\mathbf{u}}$$

Consideration of the thickness change

- Tension test and shear test of a $3-$layers composite

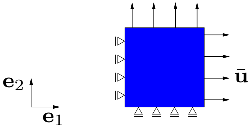




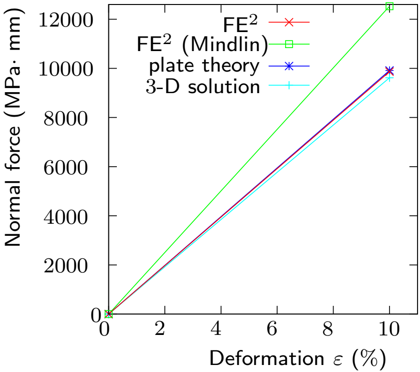
$\Rightarrow$ 7% error between FE^2 (Mindlin) and the 3-D solution

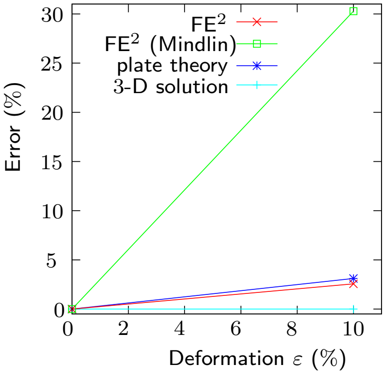
$\Rightarrow$ Exactly the same results for the FE^2 including the thickness change

Consideration of the thickness change



- Biaxial tension test of a $3-$layers composite

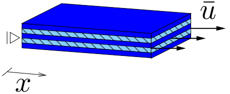




$\Rightarrow$ 30% error between FE^2 (Mindlin) and the 3-D solution

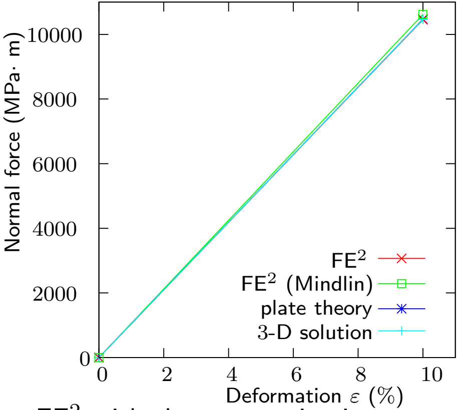
$\Rightarrow$ Accurate results for the FE^2 including the thickness change

④ Numerical results



- Tension test of a 10-layers hybrid laminate, elastic isotropic and anisotropic materials

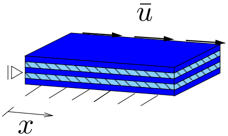
- Metal: elastic isotropic material
  $E = 70500$ MPa,
  $\nu = 0.3$
- CFRP: Transverse isotropic elastic material
  $E_1 = 138 \cdot 10^3$ MPa,
  $E_2 = 10.2 \cdot 10^3$ MPa,
  $G_{23} = 5.7 \cdot 10^3$ MPa,
  $\nu_{12} = 0.3$,
  $\nu_{23} = 0.275$

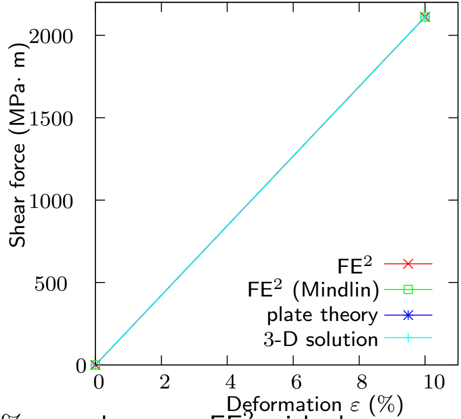


$\Rightarrow$ Less than 0.1% error between FE^2 with the new projection and the 3-D solution for a tension test with a hybrid laminate

④ Numerical results

- Shear test of a 10-layers hybrid laminate, elastic isotropic and anisotropic materials

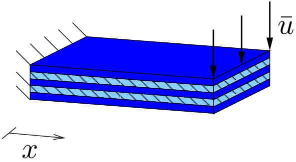


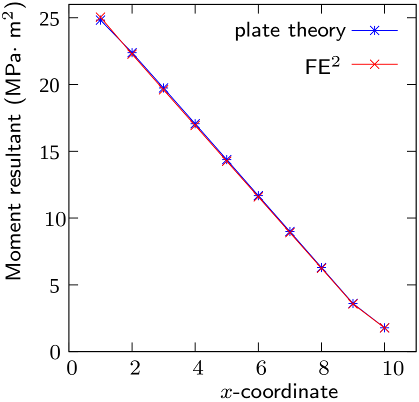


$\Rightarrow$ Less than 0.1% error between FE^2 with the new projection and the 3-D solution for a shear test with a hybrid laminate

④ Numerical results

- Bending test, 10-layers hybrid laminate, elastic isotropic and anisotropic materials

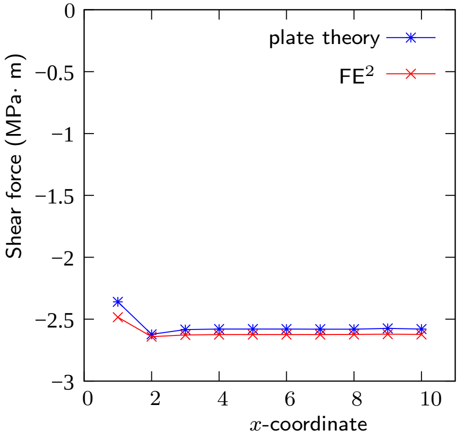




4 Numerical results

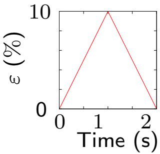
- Bending test, 10-layers hybrid laminate

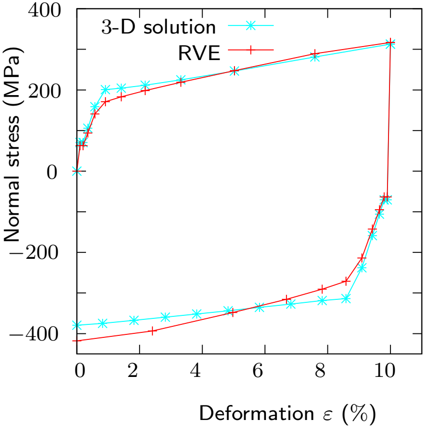
  Shear force

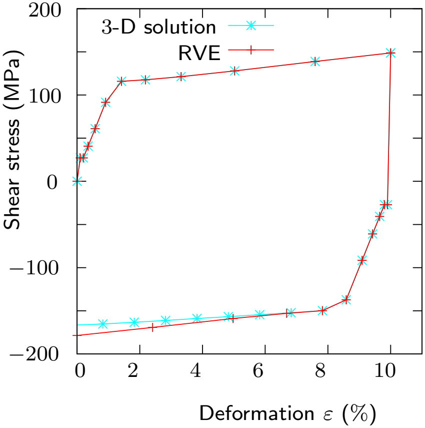


5 Numerical results

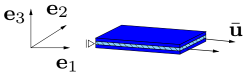
- Tension test and shear test of one elastoplastic layer



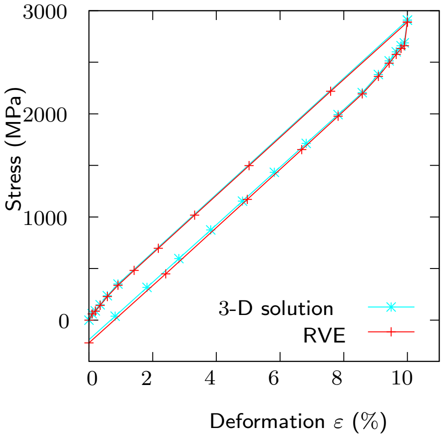




5 Numerical results

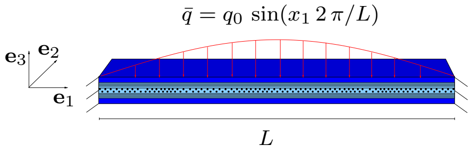


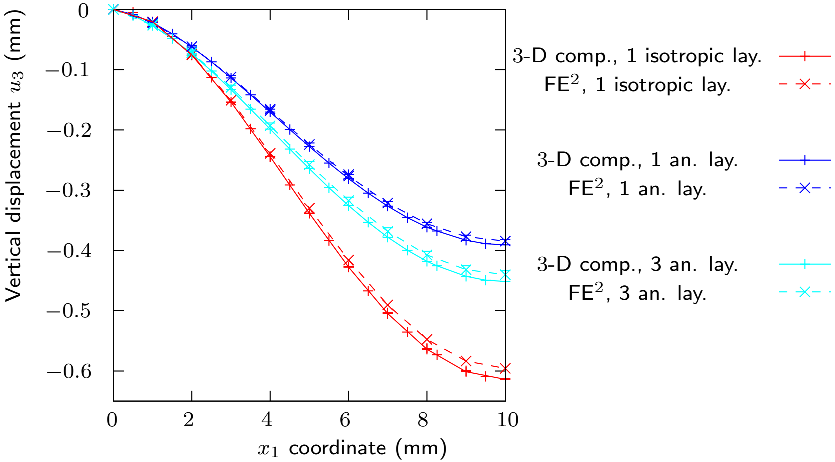
- Tension test of a three-layers sandwich plate (elastoplastic material behaviour and isotropic elastic)



⑤ Numerical results

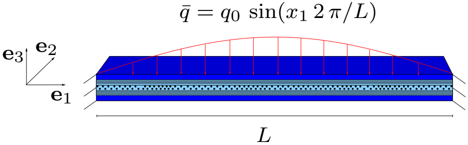
- Pagano problem, elastic isotropic and anisotropic materials

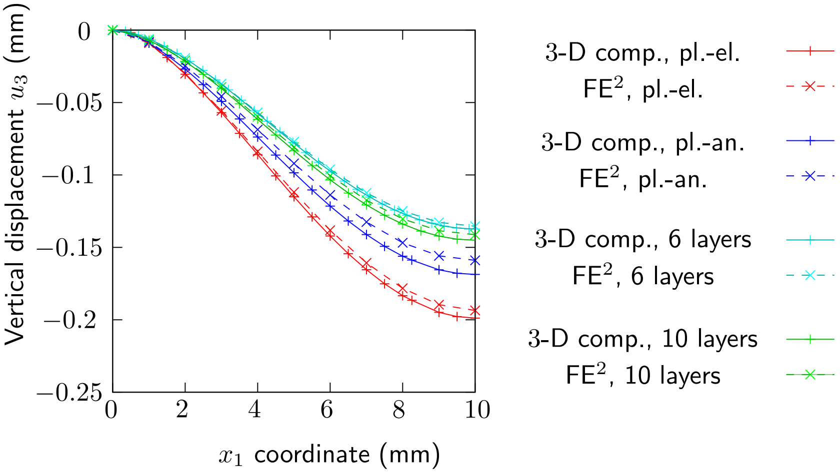




⑤ Numerical results

- Pagano problem, linear and non-linear materials





5 Conclusions and outlook

5 Conclusions

- Homogenisation principle for a thin plate
  - Plate kinematics
  - Projection of the deformations on the boundary of the RVE
  - Treatment of thickness changes
  - Solve a meso boundary problem over the RVE
  - Consistent tangent in the framework of MLNA
- Outlook
  - Fluctuations vs. zig-zag models
  - Complex material models,
    e. g. interphase between the metal and the polymer
  - Macroscopic contact: modelling of deep drawing processes

Numerical multi-scale modelling of composite plates

C. Helfen, S. Diebels

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Thank you for your attention