Thema: Basic Vector Calculus

Important technical terms and concepts: basis vectors, position vector, components of a vector, cartesian coordinates of a point, scalar product of two vectors, projection of a vector, magnitude of a vector, angle between two vectors, cross product or vector product of two vectors, scalar triple product of three vectors, line equation, explicit/implicit plane equation, normal unit vector, basis system with oblique angle, linear system of equations, linearily dependent/independent vectors

Problem 1 – solved 26.10.2018

Given a frame of reference \( \{O, e_x, e_y, e_z\} \), where \( \{e_x, e_y, e_z\} \) build an orthonormal basis. Defined are the following 3 vectors:

\[
a = \begin{pmatrix} 1 \text{ m} \\ 0 \text{ m} \\ 3 \text{ m} \end{pmatrix}, \quad b = \begin{pmatrix} -2 \text{ m} \\ 4 \text{ m} \\ 6 \text{ m} \end{pmatrix}, \quad c = \begin{pmatrix} 0 \text{ m} \\ 1 \text{ m} \\ 0 \text{ m} \end{pmatrix}
\]

1) Check, if the vectors are linearly independent.

2) Which of the vectors are normal to each other?

3) Write the vectors \( a \), \( b \) and \( c \) as position vectors of points \( A \), \( B \) and \( C \). How large is the area of the triangle \( ABC \) ?

4) Calculate the volume of the parallelepiped spanned by the three vectors?

5) Prove for arbitrary 3-dimensional vectors \( u, v, w \) the vector algebraic identity:

\[
u \times (v \times w) = v (u \cdot w) - w (u \cdot v)
\]
Problem 1 – Solution

1) By definition we have to check

\[ \alpha a + \beta b + \gamma c = 0 \implies \alpha = \beta = \gamma = 0 \]

This leads to a linear system of equations with three equations for the unknown coefficients \( \alpha, \beta \) and \( \gamma \):

\[
\begin{align*}
\alpha - 2 \beta &= 0, \\
4 \beta + \gamma &= 0, \\
3 \alpha + 6 \beta &= 0.
\end{align*}
\] (1.1)

From (1.1)_1 and (1.1)_3 we directly obtain \( \alpha = \beta = 0 \), from which (with (1.1)_2) we conclude \( \gamma = 0 \). Thus we have shown that the vectors \( a, b \) and \( c \) are linearly independent.

2) Two vectors are perpendicular if the scalar product vanishes. We compute the three scalar products:

\[
\begin{align*}
a \cdot b &= 1 \cdot (-2) m^2 + 0 \cdot 4 m^2 + 3 \cdot 6 m^2 = 16 m^2, \\
a \cdot c &= 1 \cdot 0 m^2 + 0 \cdot 1 m^2 + 3 \cdot 0 m^2 = 0 m^2, \\
b \cdot c &= -2 \cdot 0 m^2 + 4 \cdot 1 m^2 + 6 \cdot 0 m^2 = 4 m^2.
\end{align*}
\]

Consequently, we have \( a \perp c \), but \( a \not\perp b \) and \( b \not\perp c \).

3) The vectors of the edges of the triangle are

\[
s_1 = b - a = \begin{pmatrix} -3 m \\ 4 m \\ 3 m \end{pmatrix} \quad \text{and} \quad s_2 = c - a = \begin{pmatrix} -1 m \\ 1 m \\ -3 m \end{pmatrix}
\]

The area of the triangle is given by

\[
A = \frac{1}{2} |s_1 \times s_2| = \frac{1}{2} \left| \begin{pmatrix} 4 \cdot (-3) m^2 - 3 \cdot 1 m^2 \\ 3 \cdot (-1) m^2 - (-3) \cdot (-3) m^2 \\ -3 \cdot 1 m^2 - 4 \cdot (-1) m^2 \end{pmatrix} \right| = \frac{1}{2} \sqrt{225 + 144 + 1} m^2
\]

thus, finally, \( A = \frac{1}{2} \sqrt{370} m^2 \approx 9,62 m^2 \).

4) The volume \( V \) of the parallelepiped is obtained using the mixed product:

\[
V = |[a, b, c]| = |(a \times b) \cdot c|
\]

The computation: first the vector product

\[
\begin{pmatrix} 1 m \\ 0 m \\ 3 m \end{pmatrix} \times \begin{pmatrix} -2 m \\ 4 m \\ 6 m \end{pmatrix} = \begin{pmatrix} 0 \cdot 6 m^2 - 3 \cdot 4 m^2 \\ 3 \cdot (-2) m^2 - 1 \cdot 6 m^2 \\ 1 \cdot 4 m^2 - 0 \cdot (-2) m^2 \end{pmatrix} = \begin{pmatrix} -12 m^2 \\ -12 m^2 \\ 4 m^2 \end{pmatrix}
\]
and afterwards the scalar product:
\[
\begin{pmatrix}
-12 \text{ m}^2 \\
-12 \text{ m}^2 \\
4 \text{ m}^2
\end{pmatrix}
\cdot c =
\begin{pmatrix}
-12 \\
-12 \\
4
\end{pmatrix}
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
= -12 \text{ m}^3 \implies V = 12 \text{ m}^3
\]

5) With respect to the given coordinate system, the components of the vectors are as follows:
\[
u = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}, \quad v = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}, \quad w = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}
\]
Then we have
\[
v \times w = \begin{pmatrix} v_y w_z - v_z w_y \\ v_z w_x - v_x w_z \\ v_x w_y - v_y w_x \end{pmatrix}
\]
and obtain
\[
u \times (v \times w) = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \times \begin{pmatrix} v_y w_z - v_z w_y \\ v_z w_x - v_x w_z \\ v_x w_y - v_y w_x \end{pmatrix} = \begin{pmatrix} u_y (v_x w_y - v_y w_x) - u_z (v_z w_y - v_y w_z) \\ u_z (v_y w_z - v_z w_y) - u_x (v_x w_y - v_y w_x) \\ u_x (v_y w_z - v_z w_y) - u_y (v_x w_y - v_y w_x) \end{pmatrix}
\]

Simply alternatively written (adding the red terms that identically vanish within each component) the result of (1.2) reads
\[
u \times (v \times w) = \begin{pmatrix} v_x (u_x w_x + u_y w_y + u_z w_z) - u_x (v_x v_y + u_y v_y + u_z v_z) \\ v_y (u_x w_x + u_y w_y + u_z w_z) - u_y (v_x v_y + u_y v_y + u_z v_z) \\ v_z (u_x w_x + u_y w_y + u_z w_z) - u_z (v_x v_y + u_y v_y + u_z v_z) \end{pmatrix}
\]

The right hand side of (1.2) is however equivalent to
\[
v (u \cdot w) - w (u \cdot v)
\]
Homework – Deadline: during lab course in week: 05.11. - 09.11.

Let \( O, e_x, e_y, e_z \) define a frame of reference and \( \{ e_x, e_y, e_z \} \) an orthonormal basis. The following three vectors are given (with \( p \in \mathbb{R} \))

\[
\begin{align*}
a &= \begin{pmatrix} 100 \text{ cm} \\ p \text{ m} \\ -1 \text{ m} \end{pmatrix}, &
 b &= \begin{pmatrix} 300 \text{ cm} \\ 0 \text{ \mu m} \\ -10 \text{ dm} \end{pmatrix}, &
 c &= \begin{pmatrix} 0 \text{ \mu m} \\ 1 \text{ m} \\ 10 \text{ dm} \end{pmatrix}.
\end{align*}
\]

Please note: Pay attention to the units. For the problems 1)–4) exact results should be given.

1) For which real number of \( p \) are the three vectors linearly dependent?

2) What is the value of \( p \), if the vector \( a \) is perpendicular to \( c \)?

3) Calculate the volume of the tetrahedron, which is spanned by \( a \), \( b \) and \( c \). For which value of \( p \) does the volume vanish?

4) For which values of \( p \) do \( \{ a, b, c \} \) represent a right-handed-/left-handed system?

5) The position of point \( G \) is given by the position vector \( g \). In the reference system \( \{ O, \tilde{a}, \tilde{b}, \tilde{c} \} \) with

\[
\begin{align*}
\tilde{a} &= \frac{a}{|a|}, &
 \tilde{b} &= \frac{b}{|b|}, &
 \tilde{c} &= \frac{c}{|c|}.
\end{align*}
\]

the position vector is given by

\[
g = \begin{pmatrix} 2 \text{ m} \\ 5 \text{ m} \\ 20 \text{ dm} \end{pmatrix}. \]

Determine the distances of point \( G \) to the lines along the axis of the reference system \( \{ O, e_x, e_y, e_z \} \), if \( p = \sqrt{2} \) holds. Round the result to two positions behind the decimal point. Remark: Firstly, depict the vector \( g \) in the reference system \( \{ O, e_x, e_y, e_z \} \). Secondly, calculate the distances in this corresponding reference system. Use a calculator for the determination of the values.

6) Due to the high lightweight potential, material modeling of fiber reinforced composites has become more significant in the last years. The microstructure shown in Figure H.1 is considered in the following.

The center line of the two highlighted fibers is given by the support vectors \( s \) and \( v \) and the direction vectors \( t \) and \( w \). Use a calculator for the determination of the values.
\[
\begin{pmatrix}
160 \mu m \\
98 \mu m \\
17 \mu m
\end{pmatrix}, \quad
\begin{pmatrix}
-85 \mu m \\
80 \mu m \\
-52 \mu m
\end{pmatrix}, \quad
\begin{pmatrix}
-64 \mu m \\
32 \mu m \\
24 \mu m
\end{pmatrix}, \quad
\begin{pmatrix}
92 \mu m \\
23 \mu m \\
-9 \mu m
\end{pmatrix}
\]

Calculate the distance between the fiber center lines as well as the angle between both fibers and the load direction \( e_x \).

Fig. H.1: Micro computertomografie image of a glass fiber reinforced polymer