Strain gradient and micromorphic plasticity

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CENTRE DES MATERIAUX
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Plan

1. Introduction
   - Mechanics of generalized continua
2. Strain gradient plasticity theory
   - Method of virtual power
   - Constitutive equations
   - Shearing of a laminate microstructure
3. Micromorphic plasticity theory
   - Balance and constitutive equations
   - Rate–independent plasticity
   - Shearing of a laminate microstructure
4. Discussion

Notations

Cartesian basis

$$\underline{\mathbf{A}} = A_i\, \underline{\mathbf{e}}_i, \quad \underset{\sim}{\mathbf{A}} = A_{ij}\, \underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j, \quad \underset{\approx}{\mathbf{A}} = \underset{\approx}{\mathbf{A}} = \underline{\underline{\underline{\mathbf{A}}}} = A_{ijk}\, \underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j \otimes \underline{\mathbf{e}}_k$$

tensor products

$$\underline{\mathbf{a}} \otimes \underline{\mathbf{b}} = a_i b_j\, \underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j, \quad \underset{\sim}{\mathbf{A}} \otimes \underset{\sim}{\mathbf{B}} = A_{ij} B_{kl}\, \underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j \otimes \underline{\mathbf{e}}_k \otimes \underline{\mathbf{e}}_l$$

$$\underset{\sim}{\mathbf{A}} \boxtimes \underset{\sim}{\mathbf{B}} = A_{ik} B_{jl}\, \underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j \otimes \underline{\mathbf{e}}_k \otimes \underline{\mathbf{e}}_l$$

contractions

$$\underline{\mathbf{A}} \cdot \underline{\mathbf{B}} = A_i B_i, \quad \underset{\sim}{\mathbf{A}} : \underset{\sim}{\mathbf{B}} = A_{ij} B_{ij}, \quad \underset{\approx}{\mathbf{A}} \,\vdots\, \underset{\approx}{\mathbf{B}} = A_{ijk} B_{ijk}$$

nabla operators

$$\boldsymbol{\nabla}_x = {}_{,i}\, \underline{\mathbf{e}}_i, \quad \boldsymbol{\nabla}_X = {}_{,I}\, \underline{\mathbf{E}}_I$$

$$\underline{\mathbf{u}} \otimes \boldsymbol{\nabla} = u_{i,j}\, \underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j, \quad \underset{\sim}{\boldsymbol{\sigma}} . \boldsymbol{\nabla} = \sigma_{ij,j}\, \underline{\mathbf{e}}_i$$

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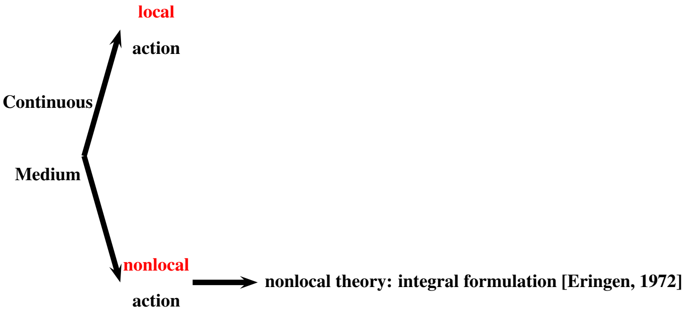
Mechanics of generalized continua

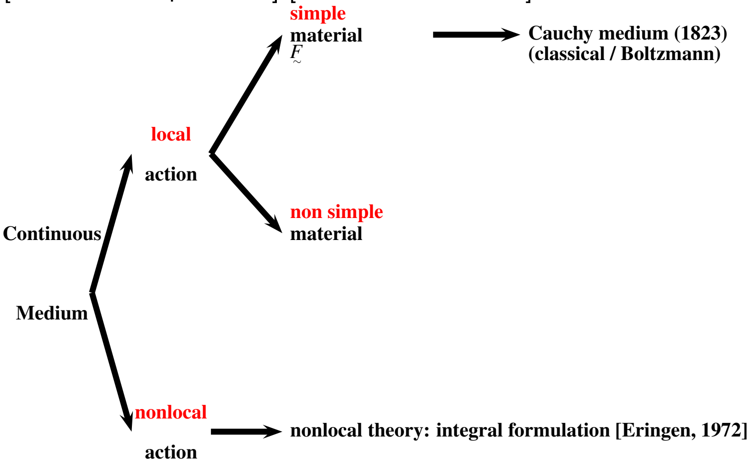
Principle of local action: *the stress state at a point $\underline{\mathbf{X}}$ depends on variables defined at this point only*

[Truesdell, Toupin, 1960] [Truesdell, Noll, 1965]



Mechanics of generalized continua

Simple material: *A material is simple at the particle $\underline{\mathbf{X}}$ if and only if its response to deformations homogeneous in a neighborhood of $\underline{\mathbf{X}}$ determines uniquely its response to every deformation at $\underline{\mathbf{X}}$.*
[Truesdell, Toupin, 1960] [Truesdell, Noll, 1965]



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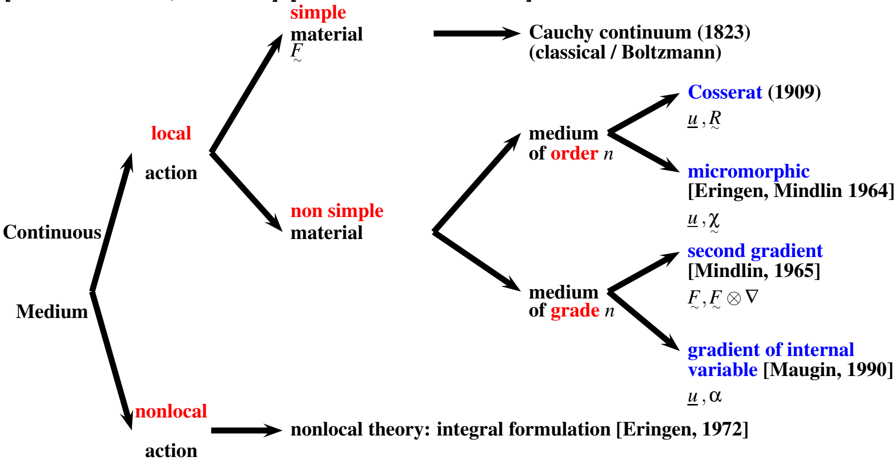

Plan

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Power of internal forces

- Model variables according to a first gradient theory

$$MODEL = \{ \quad \underline{\mathbf{v}}, \quad \underline{\mathbf{v}} \otimes \boldsymbol{\nabla}, \quad \dot{p}, \quad \boldsymbol{\nabla} \dot{p} \quad \}$$

  velocity $\underline{\mathbf{v}}$ and cumulative plastic strain p are assumed to be independent degrees of freedom

- Virtual power of internal forces of a subdomain $\mathcal{D} \subset \mathcal{B}$ of the body

$$\mathcal{P}^{(i)}(\underline{\mathbf{v}}^*, \dot{p}^*) = \int_{\mathcal{D}} p^{(i)}(\underline{\mathbf{v}}^*, \dot{p}^*)\, dV$$

  simple stress tensor $\underset{\sim}{\boldsymbol{\sigma}}$, generalized stresses a (unit MPa), $\underline{\mathbf{b}}$ (unit MPa.mm), microforces according to [Gurtin, 2002]

- The virtual power density of internal forces is a linear form on the fields of virtual modeling variables

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : (\underline{\mathbf{v}}^* \otimes \boldsymbol{\nabla}) + a\, \dot{p}^* + \underline{\mathbf{b}} \cdot \boldsymbol{\nabla} \dot{p}^*$$

- The virtual power density of internal forces is invariant with respect to superimposed rigid body motion $\Rightarrow \underset{\sim}{\boldsymbol{\sigma}}$ is symmetric [Germain, 1973a]

Power of contact forces

- Application of Gauss theorem to the power of internal forces

$$\begin{aligned}\int_{\mathcal{D}} p^{(i)}\, dV \;&=\; \int_{\partial\mathcal{D}} \underline{\mathbf{v}}^{\,*} \cdot \underset{\sim}{\boldsymbol{\sigma}} \cdot \underline{\mathbf{n}}\; dS + \int_{\partial\mathcal{D}} \dot{p}^{*}\, \underline{\mathbf{b}} \cdot \underline{\mathbf{n}}\; dS \\ &-\; \int_{\mathcal{D}} \underline{\mathbf{v}}^{\,*} \cdot \underset{\sim}{\boldsymbol{\sigma}} \cdot \boldsymbol{\nabla}\, dV - \int_{\mathcal{D}} \dot{p}^{*} \left(\underline{\mathbf{b}} \cdot \boldsymbol{\nabla} - a\right) dV\end{aligned}$$

  The form of the previous boundary integral dictates the form of the

- power of contact forces acting on the boundary $\partial\mathcal{D}$ of the subdomain $\mathcal{D} \subset \mathcal{B}$

$$\mathcal{P}^{(c)}(\underline{\mathbf{v}}^{\,*}, \dot{p}^{*}) = \int_{\partial\mathcal{D}} p^{(c)}(\underline{\mathbf{v}}^{\,*}, \dot{p}^{*})\, dS$$

$$p^{(c)}(\underline{\mathbf{v}}^{\,*}, \dot{p}^{*}) = \underline{\mathbf{t}} \cdot \underline{\mathbf{v}}^{\,*} + a_c \dot{p}^{*}$$

  simple traction $\underline{\mathbf{t}}$ (unit MPa), double traction a_c (unit MPa.mm)

Power of forces acting at a distance

$$\mathcal{P}^{(e)}(\underline{\mathbf{v}}^*, \dot{p}^*) = \int_{\mathcal{D}} p^{(e)}(\underline{\mathbf{v}}^*, \dot{p}^*)\, dV$$

$$p^{(e)}(\underline{\mathbf{v}}^*, \dot{p}^*) = \underline{\mathbf{f}} \cdot \underline{\mathbf{v}}^* + \underset{\sim}{\mathbf{c}} : (\underline{\mathbf{v}}^* \otimes \boldsymbol{\nabla}) + a_e \dot{p}^* + \underline{\mathbf{b}}_e \cdot \boldsymbol{\nabla} \dot{p}^*$$

simple body forces $\underline{\mathbf{f}}$ (unit $\mathrm{N.mm}^{-3}$), double body forces $\underset{\sim}{\mathbf{c}}$ and a_e (unit $\mathrm{N.mm}^{-2}$), triple body force $\underline{\mathbf{b}}_e$ (unit $\mathrm{N.mm}^{-1}$)

$$\begin{aligned}\mathcal{P}^{(e)}(\underline{\mathbf{v}}^*, \dot{p}^*) &= \int_{\partial\mathcal{D}} \left(\underline{\mathbf{v}} \cdot \underset{\sim}{\mathbf{c}} \cdot \underline{\mathbf{n}} + \dot{p}^* \underline{\mathbf{b}}_e \cdot \underline{\mathbf{n}}\right)\, dS \\ &- \int_{\mathcal{D}} \left(\underline{\mathbf{v}}^* \cdot (\underset{\sim}{\mathbf{c}} \cdot \boldsymbol{\nabla} - \underline{\mathbf{f}})\, dV + \dot{p}^*(\underline{\mathbf{b}}_e \cdot \boldsymbol{\nabla} - a_e)\right)\, dV\end{aligned}$$

Principle of virtual power

In the static case, $\forall \underline{\mathbf{v}}^*, \forall \dot{p}^*, \forall \mathcal{D} \subset \mathcal{B}$,

$$\mathcal{P}^{(i)}(\underline{\mathbf{v}}^*, \dot{p}^*) = \mathcal{P}^{(c)}(\underline{\mathbf{v}}^*, \dot{p}^*) + \mathcal{P}^{(e)}(\underline{\mathbf{v}}^*, \dot{p}^*)$$

[Germain, 1973b]

Principle of virtual power

In the static case, $\forall \underline{\mathbf{v}}^*, \forall \dot{p}^*, \forall \mathcal{D} \subset \mathcal{B}$,

$$\mathcal{P}^{(i)}(\underline{\mathbf{v}}^*, \dot{p}^*) = \mathcal{P}^{(c)}(\underline{\mathbf{v}}^*, \dot{p}^*) + \mathcal{P}^{(e)}(\underline{\mathbf{v}}^*, \dot{p}^*)$$

which leads to

$$\int_{\partial\mathcal{D}} \underline{\mathbf{v}}^* \cdot (\underline{\mathbf{t}} - (\underset{\sim}{\boldsymbol{\sigma}} - \underline{\mathbf{c}}) \cdot \underline{\mathbf{n}}) + \dot{p}^*(a_c - (\underline{\mathbf{b}} - \underline{\mathbf{b}}^e) \cdot \underline{\mathbf{n}})\, dS$$

$$+ \int_{\mathcal{D}} \underline{\mathbf{v}}^* \cdot ((\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\mathbf{c}}) \cdot \boldsymbol{\nabla} + \underline{\mathbf{f}}) + \dot{p}^*((\underline{\mathbf{b}} - \underline{\mathbf{b}}_e) \cdot \boldsymbol{\nabla} - a + a_e)\, dV = 0$$

cf. [Germain, 1973b, Forest and Sievert, 2003, Kirchner and Steinmann, 2005, Lazar and Maugin, 2007, Hirschberger et al., 2007]

Balance and boundary conditions

The application of the principle of virtual power leads to the

- balance of momentum equation (static case)

$$(\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\mathbf{c}}) \cdot \boldsymbol{\nabla} + \underline{\mathbf{f}} = 0, \quad \forall \underline{\mathbf{x}} \in \mathcal{B}$$

- balance of generalized moment of momentum equation (static case)

$$(\underline{\mathbf{b}} - \underline{\mathbf{b}}_e) \cdot \boldsymbol{\nabla} - a + a_e = 0, \quad \forall \underline{\mathbf{x}} \in \mathcal{B}$$

- boundary conditions

$$(\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\mathbf{c}}) \cdot \underline{\mathbf{n}} = \underline{\mathbf{t}}, \quad \forall \underline{\mathbf{x}} \in \partial\mathcal{B}$$

$$(\underline{\mathbf{b}} - \underline{\mathbf{b}}_e) \cdot \underline{\mathbf{n}} = a_c, \quad \forall \underline{\mathbf{x}} \in \partial\mathcal{B}$$

Plan

Continuum thermodynamics

- Enhance the local balance of energy and the entropy inequality

$$\rho\dot{\epsilon} = p^{(i)} - \mathrm{div}\,\underline{\mathbf{q}} + \rho r, \quad -\rho(\dot{\psi} + \eta\dot{T}) + p^{(i)} - \frac{\underline{\mathbf{q}}}{T}.\nabla T \geq 0$$

- Decomposition of total strain

$$\underline{\mathbf{v}} = \underline{\dot{\mathbf{u}}}, \quad \underset{\sim}{\varepsilon} = \frac{1}{2}(\underline{\mathbf{u}} \otimes \nabla + \nabla \otimes \underline{\mathbf{u}})$$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}^e + \underset{\sim}{\varepsilon}^p$$

- Consider the constitutive functionals:

$$\begin{aligned}
\psi &= \hat{\psi}(\underset{\sim}{\varepsilon}^e, T, p, \alpha, \nabla p), \; \eta = \hat{\eta}(\underset{\sim}{\varepsilon}^e, T, p, \alpha, \nabla p) \\
\underset{\sim}{\sigma} &= \hat{\underset{\sim}{\sigma}}(\underset{\sim}{\varepsilon}^e, T, p, \alpha, \nabla p) \\
a &= \hat{a}(\underset{\sim}{\varepsilon}^e, T, p, \alpha, \nabla p), \quad \underline{\mathbf{b}} = \hat{\underline{\mathbf{b}}}(\underset{\sim}{\varepsilon}^e, T, p, \alpha, \nabla p)
\end{aligned}$$

State laws

- Clausius–Duhem inequality (isothermal)

$$(\underset{\sim}{\boldsymbol{\sigma}} - \rho \frac{\partial \psi}{\partial \underset{\sim}{\boldsymbol{\varepsilon}}^e}) : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p + (a - \rho \frac{\partial \psi}{\partial p}) \dot{p} + (\underline{\mathbf{b}} - \rho \frac{\partial \psi}{\partial \boldsymbol{\nabla} p}) \cdot \boldsymbol{\nabla} \dot{p} + \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p - \rho \frac{\partial \psi}{\partial \alpha} \dot{\alpha} \geq 0$$

- Derive the state laws [Coleman and Noll, 1963]

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho \frac{\partial \hat{\psi}}{\partial \underset{\sim}{\boldsymbol{\varepsilon}}^e}, \quad X = \rho \frac{\partial \hat{\psi}}{\partial \alpha}, \quad R = \frac{\partial \hat{\psi}}{\partial p}, \quad \underline{\mathbf{b}} = \frac{\partial \hat{\psi}}{\partial \boldsymbol{\nabla} p}$$

- Residual dissipation

$$D^{res} = \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p + (a - R) \dot{p} - X \dot{\alpha} \geq 0$$

Flow rule and evolution law

- Introduce an equivalent stress measure σ_{eq} such that

$$\underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p = \sigma_{eq} \dot{p}$$

which defines the cumulative plastic strain rate $\dot{p}$

$$D^{res} = (\sigma_{eq} + a - R)\dot{p} - X\dot{\alpha} \geq 0$$

- Introduce the viscoplastic potential $\Omega(\sigma_{eq} + a - R, X)$ such that

$$\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p = \frac{\partial \Omega}{\partial(\sigma_{eq} + a - R)}, \quad \dot{\alpha} = -\frac{\partial \Omega}{\partial X}$$

Ω convex with respect to the first, concave with respect to the second variable for positive dissipation

- Rate–independent case; introduce the yield function

$$f(\underset{\sim}{\boldsymbol{\sigma}}, a, R) = \sigma_{eq} - R_0 - R + a$$

where R_0 is the initial yield stress

$$\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p = \dot{p} \frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\sigma}}}, \quad \sigma_{eq} = \underset{\sim}{\boldsymbol{\sigma}} : \frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\sigma}}}$$

Specific constitutive equations

- Free energy function

$$\rho\psi(\underset{\sim}{\varepsilon}^e, p, \nabla p) = \frac{1}{2}\underset{\sim}{\varepsilon}^e : \underset{\approx}{\mathbf{\Lambda}} : \underset{\sim}{\varepsilon}^e + \frac{1}{2}Hp^2 + \frac{1}{2}\nabla p \cdot \underset{\sim}{\mathbf{A}} \cdot \nabla p$$

- State laws

$$\underset{\sim}{\sigma} = \underset{\approx}{\mathbf{\Lambda}} : \underset{\sim}{\varepsilon}^e, \quad R = Hp, \quad \underline{\mathbf{b}} = \underset{\sim}{\mathbf{A}} \cdot \nabla p$$

- Balance of generalized momentum (homogeneous material)

$$a = \operatorname{div} \underline{\mathbf{b}} = \operatorname{div}(\underset{\sim}{\mathbf{A}} \cdot \nabla p) = \underset{\sim}{\mathbf{A}} : (\nabla \otimes \nabla p)$$

- Yield function $\quad f(\underset{\sim}{\sigma}, R, a) = J_2(\underset{\sim}{\sigma}) - R_0 - Hp + a$
- Consistency condition; $\dot{p}$ is solution of a p.d.e.

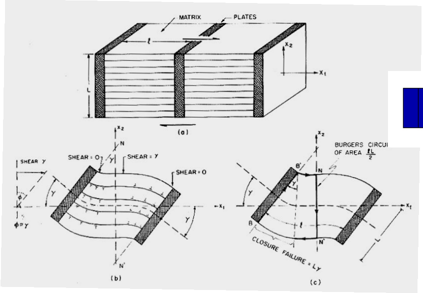
$$(H + \frac{\partial f}{\partial \underset{\sim}{\sigma}} : \underset{\approx}{\mathbf{\Lambda}} : \frac{\partial f}{\partial \underset{\sim}{\sigma}})\dot{p} + \underset{\sim}{\mathbf{A}} : (\nabla \otimes \nabla \dot{p}) = \frac{\partial f}{\partial \underset{\sim}{\sigma}} : \underset{\approx}{\mathbf{\Lambda}} : \dot{\underset{\sim}{\varepsilon}}$$

- Isotropic case + von Mises : Aifantis model [Aifantis, 1987]

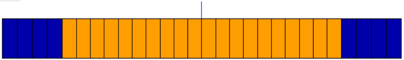
$$\underset{\sim}{\mathbf{A}} = c^2 \underset{\sim}{\mathbf{1}}, \quad \sigma_{eq} = R_0 + Hp - c^2 \Delta p$$

Plan

Confined plasticity



[Ashby, 1970]



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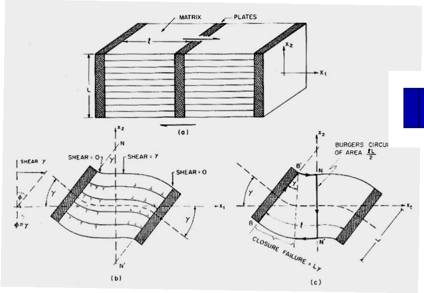
periodic simple shear test:
classical solution

conventional continuum plasticity predicts homogenenous plastic deformation in the layers

Confined plasticity



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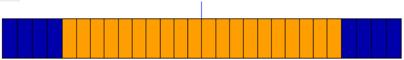

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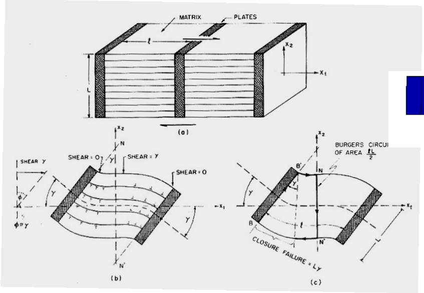

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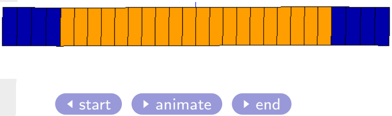

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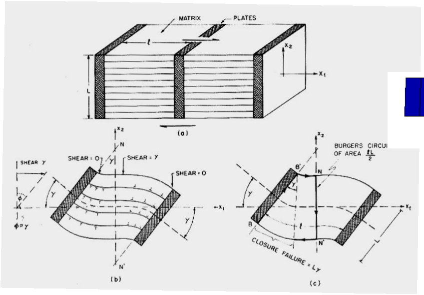

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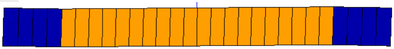

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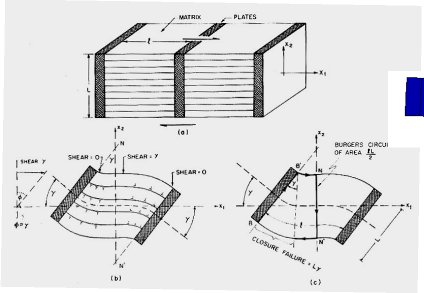

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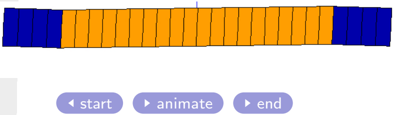

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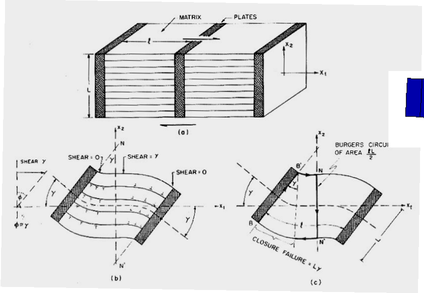

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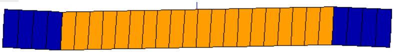

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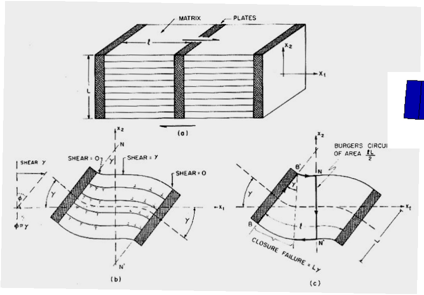

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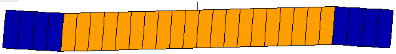

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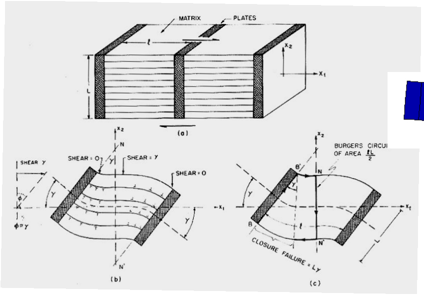

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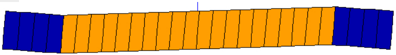

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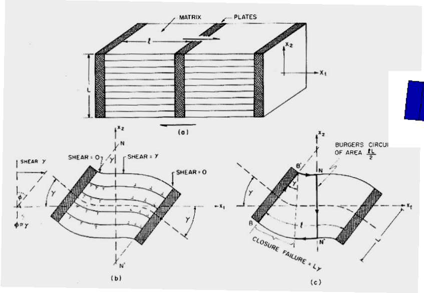

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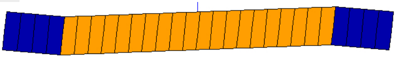

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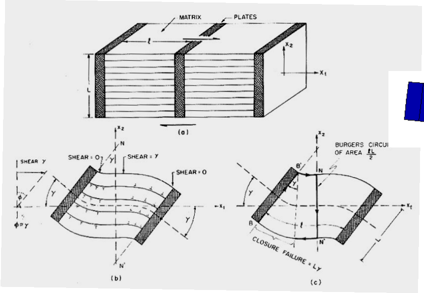

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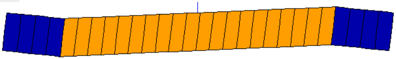

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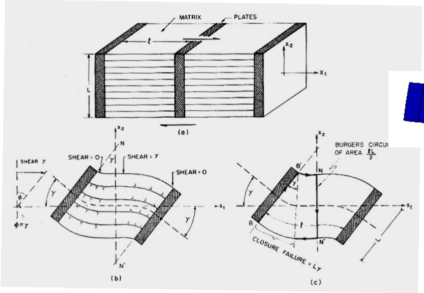

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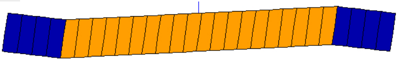

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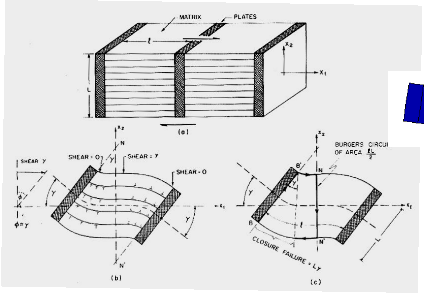

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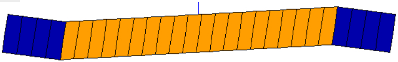

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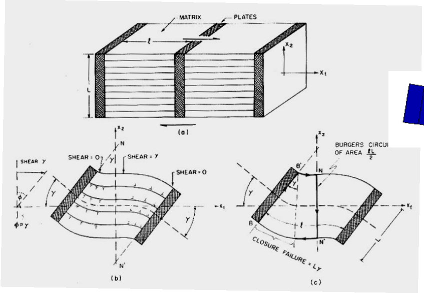

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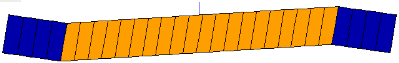

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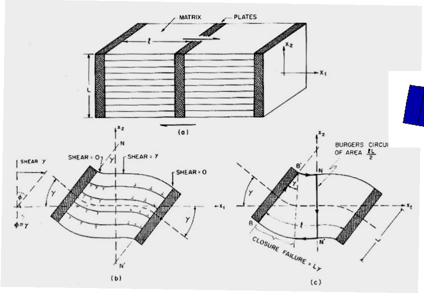
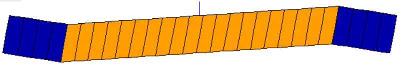


[Ashby, 1970]



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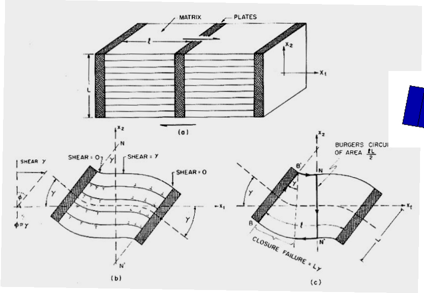

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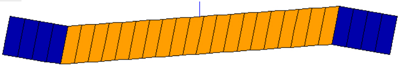

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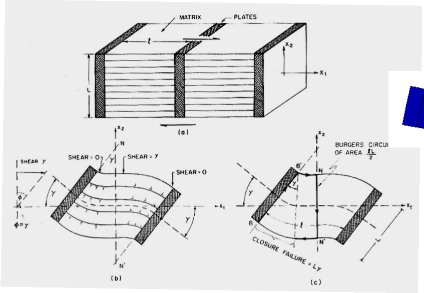

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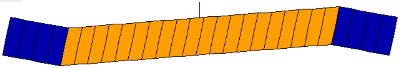

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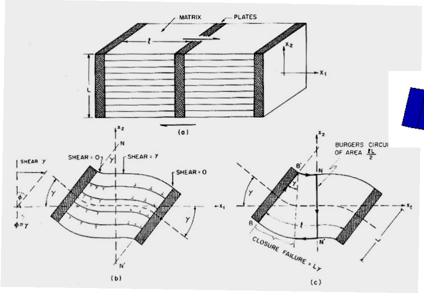

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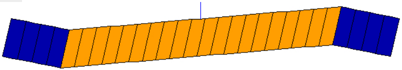

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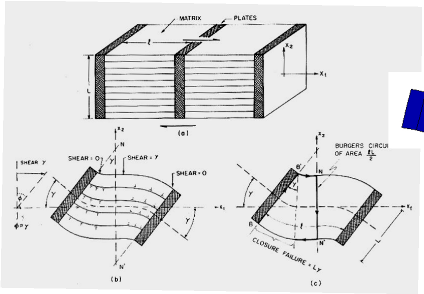

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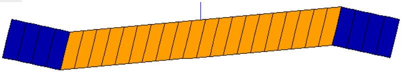

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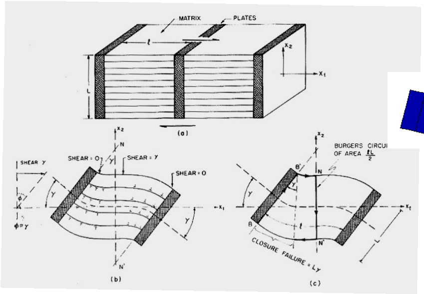

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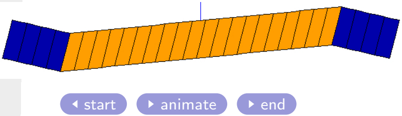

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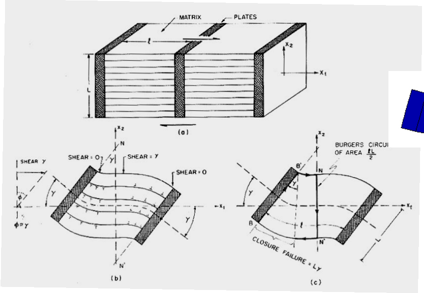

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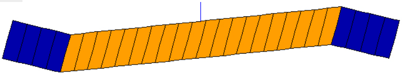

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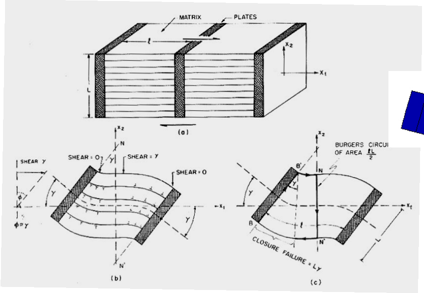

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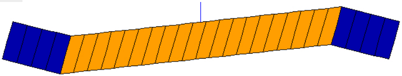

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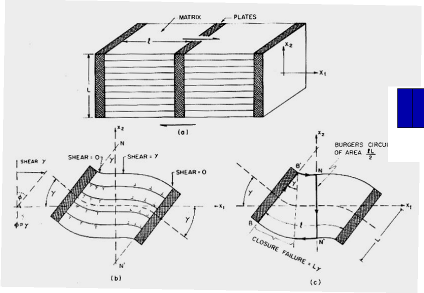

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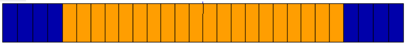

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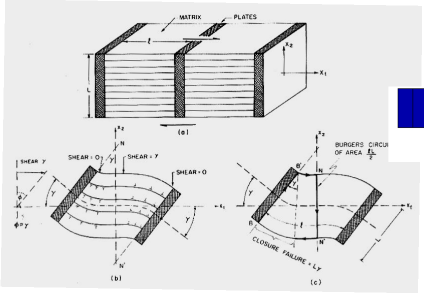

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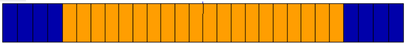

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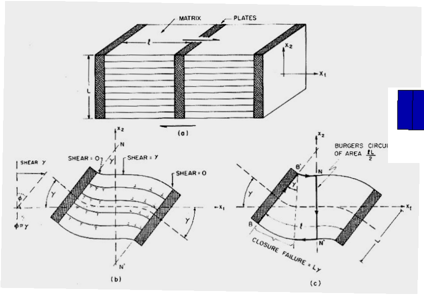

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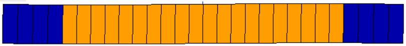

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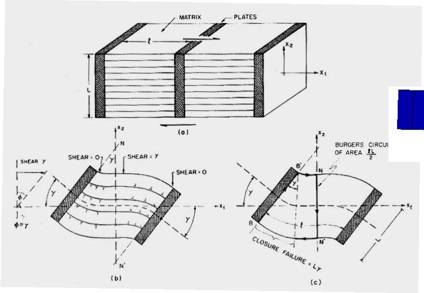

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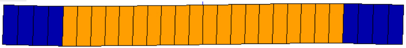

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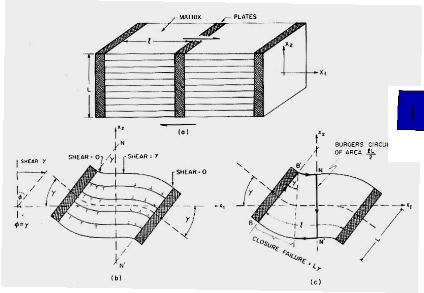

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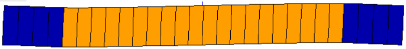

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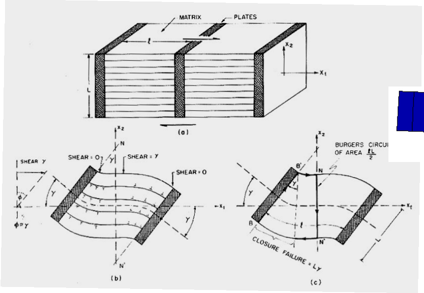

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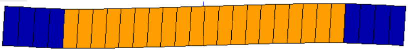

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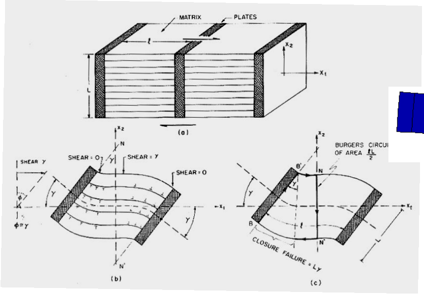

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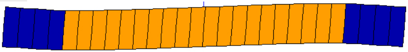

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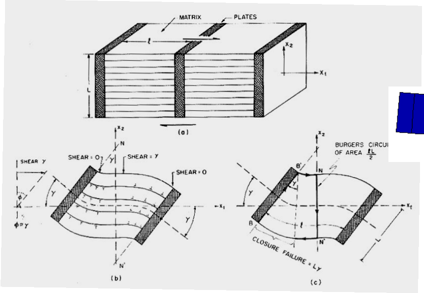

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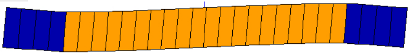

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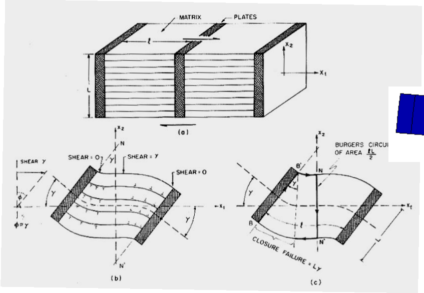

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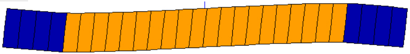

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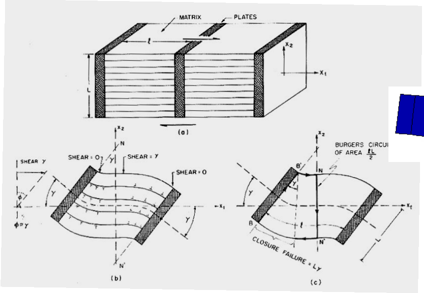

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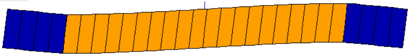

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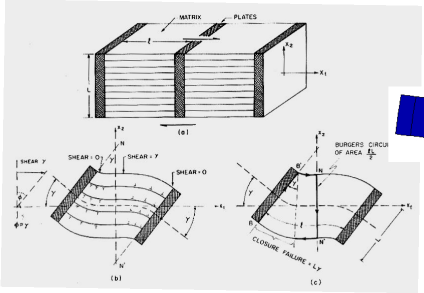

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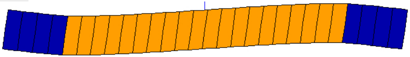

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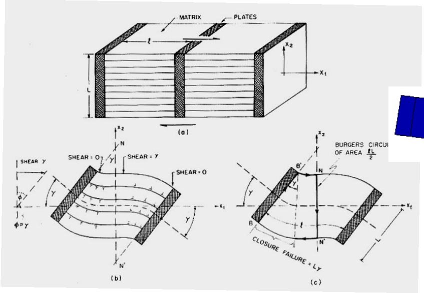

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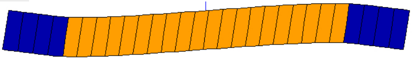

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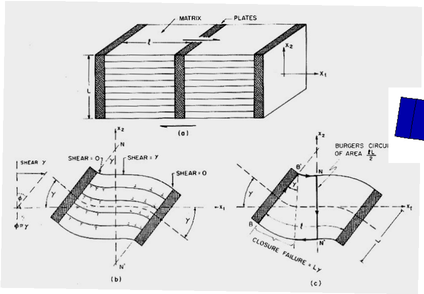

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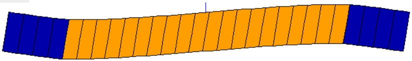

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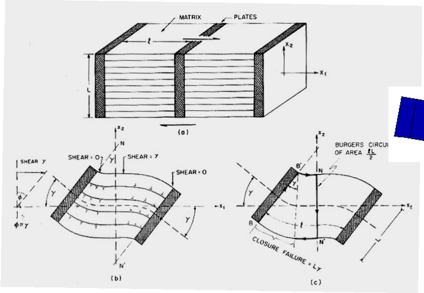

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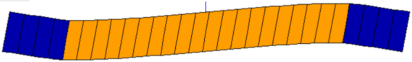

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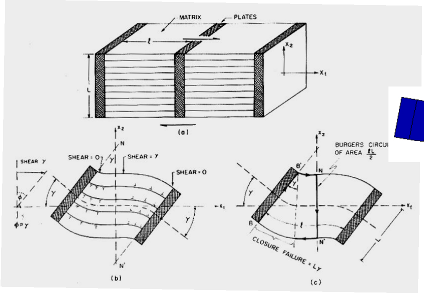

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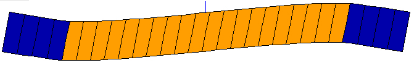

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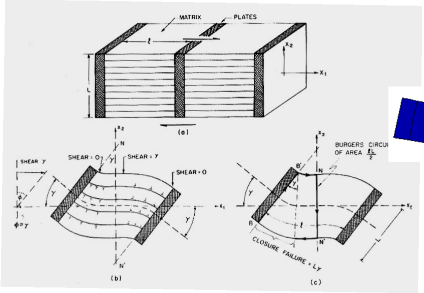

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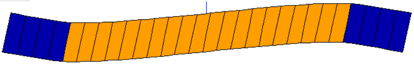

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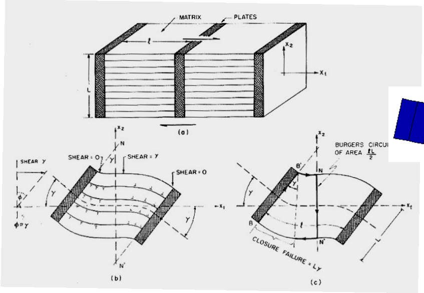

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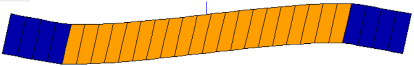

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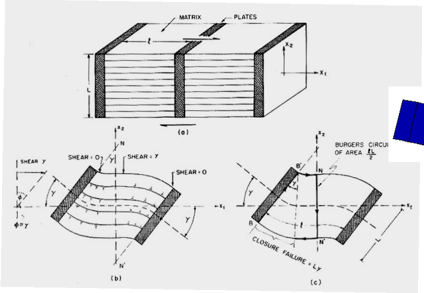

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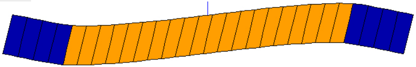

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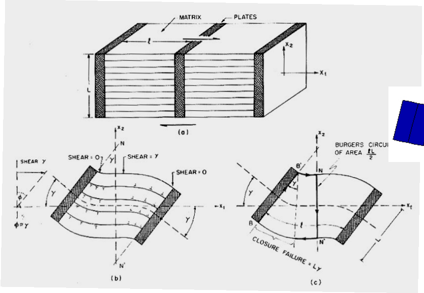

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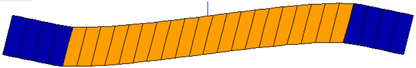

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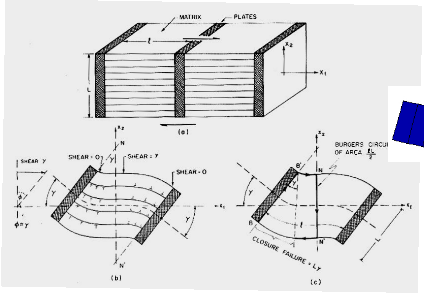

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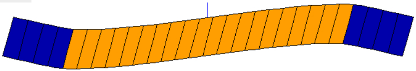

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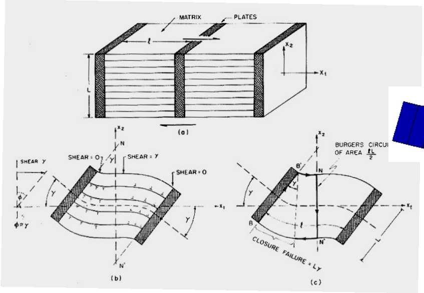

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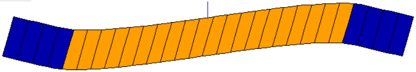

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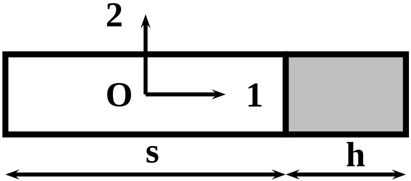
Laminate microstructure under shear

Unit cell of a periodic two–phase laminate $\quad l = s + h$



Aifantis material in the white (soft) phase, purely elastic gray (hard) phase

- Form of the solution for imposed mean shear $\bar{\gamma}$

$$u_1 = \bar{\gamma}\, x_2, \quad u_2(x_1) = u(x_1), \quad u_3 = 0$$

  unknown periodic functions $u(x_1), p(x_1)$

- Deformation gradient and strain

$$[\nabla \underline{u}] = \begin{bmatrix} 0 & \bar{\gamma} & 0 \\ u_{,1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\underset{\sim}{\varepsilon}] = \begin{bmatrix} 0 & \frac{1}{2}(\bar{\gamma} + u_{,1}) & 0 \\ \frac{1}{2}(\bar{\gamma} + u_{,1}) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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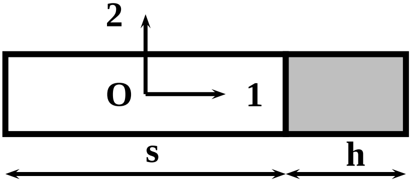

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Resolution of the b.v.p.

Let us consider homogeneous isotropic elasticity and no hardening in the plastic phase for simplicity

- Equilibrium: homogeneous shear stress σ_{12} throughout the laminate
- Displacement in the hard phase

$$\sigma_{12} = \mu(\bar{\gamma} + u^h_{,1}) \quad \Longrightarrow \quad u^h_{,1} = C, \quad u^h = C x_1 + D$$

- Plastic strain in the soft phase

$$\dot{\underset{\sim}{\varepsilon}}^p = \frac{3}{2}\dot{p}\frac{\underset{\sim}{\mathbf{s}}}{J_2(\underset{\sim}{\sigma})}, \quad \dot{\underset{\sim}{\varepsilon}}^p = \frac{\sqrt{3}}{2}\dot{p}(\underline{\mathbf{e}}_1 \otimes \underline{\mathbf{e}}_2 + \underline{\mathbf{e}}_2 \otimes \underline{\mathbf{e}}_1)$$

from the yield condition we get

$$\sigma_{12} = \mu(\bar{\gamma} + u^h_{,1}) \quad \Longrightarrow \quad u^h_{,1} = C, \quad u^h = C x_1 + D$$

so that the plastic strain is parabolic

$$p = \alpha(x_1^2 - \frac{s^2}{4})$$

- Continuity of plastic strain at the interface $\qquad p(\pm s/2) = 0$

Resolution of the b.v.p.

- Displacement in the soft phase

$$\sigma_{12} = \mu(\bar{\gamma} + u^s_{,1} - \sqrt{3}p) \quad \Longrightarrow \quad u^s_{,1} = C + \sqrt{3}p$$

$$u^s = (C - \alpha\sqrt{3}\frac{s^2}{4})x_1 + \alpha\frac{\sqrt{3}}{3}x_1^3$$

Interface conditions

- Displacement continuity at $x_1 = \pm s/2$

$$u^s(\frac{s}{2}) = u^h(\frac{s}{2}) \quad \Longrightarrow \quad -\sqrt{3}\alpha\frac{s^3}{12} = D$$

- Displacement periodicity at $x_1 = -s/2$ and $x_1 = s/2 + h$

$$u^s(-\frac{s}{2}) = u^h(\frac{s}{2} + h) \quad \Longrightarrow \quad \sqrt{3}\alpha\frac{s^3}{12} = Cl + D$$

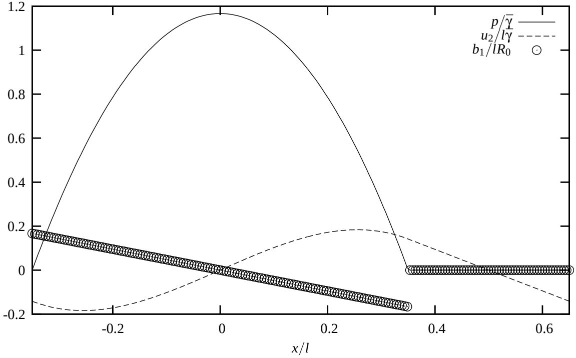
- Continuity of the stress vector at $x_1 = \pm s/2$

$$R_0 - 2c\alpha = \mu\sqrt{3}(\bar{\gamma} + C)$$

- The wanted constants are deduced from the previous equations

$$C = \frac{R_0 - \sqrt{3}\mu\bar{\gamma}}{\sqrt{3}\mu + \dfrac{12cl}{\sqrt{3}s^3}}, \quad D = -C\frac{l}{2}, \quad \alpha = -\frac{12}{\sqrt{3}}\frac{D}{s^3}$$

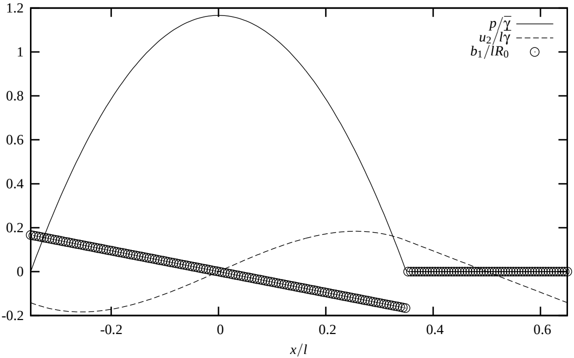
Plastic strain profile in the channel



μ (MPa)	R_0 (MPa)	c (MPa.mm^2)	f	l (μm)	$\bar{\gamma}$
30000	20	0.005	0.7	10	0.01

- Characteristic length: $l_c = \sqrt{c/\mu} = 0.4\ \mu$m, leading to strong size effects in the micron range and below

Plastic strain profile in the channel



μ (MPa)	R_0 (MPa)	c (MPa.mm^2)	f	l (μm)	$\bar{\gamma}$
30000	20	0.005	0.7	10	0.01

- The higher order stress $b_1 = 2c\alpha$ experiences a jump at the interface $s = \pm s/2$:

$$b_1(\frac{s^+}{2}) - b_1(\frac{s^-}{2}) = 0 - c\alpha s, \quad [\![b_1]\!](\frac{s}{2}) = -c\alpha s$$

Overall size effect

- Macroscopic stress strain relation

$$\frac{\sigma_{12}}{\mu} = \frac{1}{\mu f s^2 + 4c} \left(\frac{\sqrt{3}}{3} f s^2 R_0 + 4c\bar{\gamma} \right)$$

bilinear response depending explicitly on channel size s

- Macroscopic stress vs mean plastic strain;

$$\bar{p} = \frac{1}{l} \int_{-s/2}^{s/2} p(x_1)\, dx_1 \quad \Longrightarrow \quad \sqrt{3}\bar{p} = f\bar{\gamma} - C(1-f) - f\frac{\sigma_{12}}{\mu}$$

$$\sigma_{12} = \frac{R_0}{\sqrt{3}} + \frac{4\sqrt{3}c}{f^3 l^2}\bar{p}$$

microstructure–induced linear hardening depending on unit cell size l

- Limit cases
  - ⋆ thick channels: size independent threshold $\sigma_{12} = R_0/\sqrt{3}$
  - ⋆ thin films: scaling law $\sigma_{12}/\bar{p} \sim 1/l^2$

Plan

1. Introduction
   - Mechanics of generalized continua
2. Strain gradient plasticity theory
   - Method of virtual power
   - Constitutive equations
   - Shearing of a laminate microstructure
3. Micromorphic plasticity theory
   - Balance and constitutive equations
   - Rate–independent plasticity
   - Shearing of a laminate microstructure
4. Discussion

Plan

General scalar microstrain gradient plasticity

- Classical and generalized plasticity

  $$DOF = \{\underline{\mathbf{u}}, \quad p_\chi\} \quad STATE = \{\underset{\sim}{\varepsilon}^e, \quad p, \quad \alpha, \quad p_\chi, \quad \nabla p_\chi\}$$

  scalar plastic microstrain variable p_χ

- Enhanced power of internal forces and extra balance equation

  $$p^{(i)} = \underset{\sim}{\sigma} : \dot{\underset{\sim}{\varepsilon}} + a\ \dot{p}_\chi + \underline{\mathbf{b}} . \nabla\ \dot{p}_\chi, \quad p^{(c)} = \underline{\mathbf{t}} . \dot{\underline{\mathbf{u}}} + a^c\ \dot{p}_\chi$$

  $$\mathrm{div}\, \underline{\mathbf{b}} - a = 0, \quad \forall \underline{\mathbf{x}} \in \Omega, \quad \underline{\mathbf{b}} . \underline{\mathbf{n}} = a^c, \quad \forall \underline{\mathbf{x}} \in \partial\Omega$$

General scalar microstrain gradient plasticity

- State laws

$$\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}^e + \underset{\sim}{\varepsilon}^p$$

$$\underset{\sim}{\sigma} = \rho \frac{\partial \psi}{\partial \underset{\sim}{\varepsilon}^e}, \quad R = \rho \frac{\partial \psi}{\partial p}, \quad X = \rho \frac{\partial \psi}{\partial \alpha}$$

$$a = \rho \frac{\partial \psi}{\partial p_\chi} + a^v, \quad \underline{\mathbf{b}} = \rho \frac{\partial \psi}{\partial \boldsymbol{\nabla} p_\chi}$$

- Evolution laws $\qquad D^{res} = \underset{\sim}{\sigma} : \dot{\underset{\sim}{\varepsilon}}^p + (a^v - R)\dot{p} - X\dot{\alpha} \geq 0$

$$\dot{\underset{\sim}{\varepsilon}}^p = \dot{\lambda} \frac{\partial f}{\partial \underset{\sim}{\sigma}}, \quad \dot{p} = -\dot{\lambda} \frac{\partial f}{\partial R}, \quad \dot{\alpha} = -\dot{\lambda} \frac{\partial f}{\partial X}$$

[Forest, 2009]

Explicit constitutive equations

- Quadratic free energy potential

$$\rho\psi(\underset{\sim}{\varepsilon}^e, p, p_\chi, \nabla p_\chi) = \frac{1}{2}\underset{\sim}{\varepsilon}^e : \underset{\approx}{\mathbf{\Lambda}} : \underset{\sim}{\varepsilon}^e + \frac{1}{2}Hp^2 + \frac{1}{2}H_\chi(p - p_\chi)^2 + \frac{1}{2}\nabla p_\chi . \underset{\sim}{\mathbf{A}} . \nabla p_\chi$$

[Forest, 2009, Dimitrijevic and Hackl, 2011]

- Constitutive equations

$$\underset{\sim}{\sigma} = \underset{\approx}{\mathbf{\Lambda}} : \underset{\sim}{\varepsilon}^e,\ a = -H_\chi(p - p_\chi),\ \underline{\mathbf{b}} = \underset{\sim}{\mathbf{A}} . \nabla p_\chi,\ R = (H + H_\chi)p - H_\chi p_\chi$$

- Substitution of constitutive into extra balance equations

$$p_\chi - \frac{1}{H_\chi}\mathrm{div}\,(\underset{\sim}{\mathbf{A}} . \nabla p_\chi) = p$$

- Homogeneous and isotropic materials $\underset{\sim}{\mathbf{A}} = A\underset{\sim}{\mathbf{1}}$

$$p_\chi - \frac{A}{H_\chi}\Delta p_\chi = p, \quad \text{b.c.} \quad \underline{\mathbf{b}} \cdot \underline{\mathbf{n}} = A\nabla p_\chi . \underline{\mathbf{n}} = a^c$$

same p.d.e. as in the *implicit gradient–enhanced elastoplasticity* with $a^c = 0$ [Engelen et al., 2003]

Link to Aifantis strain gradient plasticity

- Yield function

$$f(\underset{\sim}{\boldsymbol{\sigma}}, R) = \sigma_{eq} - R_0 - R$$

- Hardening law

$$R = \frac{\partial \psi}{\partial p} = (H + H_\chi)p - H_\chi p_\chi$$

- Under plastic loading

$$\sigma_{eq} = R_0 + H p_\chi - A(1 + \frac{H}{H_\chi})\Delta p_\chi$$

compare with Aifantis model [Aifantis, 1987]

$$\sigma_{eq} = R_0 + R(p) - c^2 \Delta p$$

The equivalence is obtained for $H_\chi = \infty$ (internal constraint):

$$p_\chi \simeq p, \quad A = c^2$$

Plan

Consistency condition

- Consistency condition

$$\begin{aligned} \dot{f} &= \frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\sigma}}} : \dot{\underset{\sim}{\boldsymbol{\sigma}}} + + \frac{\partial f}{\partial R} \dot{R} \\ &= \frac{\partial \sigma_{eq}}{\partial \sigma} : \underset{\approx}{\boldsymbol{\Lambda}} : (\dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}^p) - \frac{\partial R}{\partial p} \dot{p} - \frac{\partial R}{\partial p_\chi} \dot{p}_\chi = 0 \end{aligned}$$

- Plastic multiplier

$$\dot{p} = \frac{\underset{\sim}{\mathbf{N}} : \underset{\approx}{\boldsymbol{\Lambda}} : \dot{\underset{\sim}{\varepsilon}} - \dfrac{\partial R}{\partial p_\chi} \dot{p}_\chi}{\underset{\sim}{\mathbf{N}} : \underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\mathbf{N}} + \dfrac{\partial R}{\partial p}}, \quad \text{with} \quad \underset{\sim}{\mathbf{N}} = \frac{\partial \sigma_{eq}}{\partial \underset{\sim}{\boldsymbol{\sigma}}}$$

  where $\dot{\underset{\sim}{\varepsilon}}$ and $\dot{p}_\chi$ are controllable variable.

- Even though the yield condition can be written as a partial differential equation, there is no need for a variational formulation of the consistency condition contrary to [Mühlhaus and Aifantis, 1991, Liebe et al., 2001]. There is no need for a plastic front tracking technique. The plastic microstrain p_χ and the generalized traction $\underline{\mathbf{b}} \, . \underline{\mathbf{n}}$ are continuous across the elastic/plastic domain.

Thermal effects

- For temperature dependent parameters

$$a = \mathrm{div}\, \underline{\mathbf{b}} = \mathrm{div}\left(A \boldsymbol{\nabla} p_\chi\right) = A \Delta p_\chi + \frac{\partial A}{\partial T} \boldsymbol{\nabla} T . \boldsymbol{\nabla} p_\chi$$

$$p_\chi - \frac{A}{H_\chi} \Delta p_\chi - \frac{1}{H_\chi} \frac{\partial A}{\partial T} \boldsymbol{\nabla} T . \boldsymbol{\nabla} p_\chi = p$$

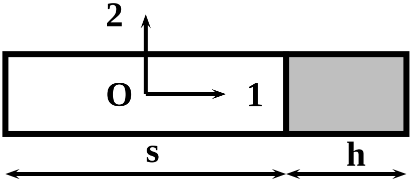
- Consistency condition

$$\dot{p} = \frac{\underset{\sim}{\mathbf{N}} : \underset{\approx}{\mathbf{\Lambda}} : (\dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}^{th}) - \dfrac{\partial R}{\partial p_\chi} \dot{p}_\chi - \dfrac{\partial R}{\partial T} \dot{T}}{\underset{\sim}{\mathbf{N}} : \underset{\approx}{\mathbf{\Lambda}} : \underset{\sim}{\mathbf{N}} + \dfrac{\partial R}{\partial p}}$$

Plan

Laminate microstructure under shear

Unit cell of a periodic two–phase laminate $l = s + h$



Micromorphic material in the white (soft) phase, purely elastic micromorphic gray (hard) phase

- Form of the solution for impose mean shear $\bar{\gamma}$

$$u_1 = \bar{\gamma}\, x_2, \quad u_2(x_1) = u(x_1), \quad u_3 = 0$$

  unknown periodic functions $u(x_1), p(x_1), p_\chi(x_1)$

- Deformation gradient and strain

$$[\nabla \underline{\mathbf{u}}] = \begin{bmatrix} 0 & \bar{\gamma} & 0 \\ u_{,1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\underset{\sim}{\varepsilon}] = \begin{bmatrix} 0 & \frac{1}{2}(\bar{\gamma} + u_{,1}) & 0 \\ \frac{1}{2}(\bar{\gamma} + u_{,1}) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Resolution of the b.v.p.

Let us consider homogeneous isotropic elasticity, homogeneous H_χ and no hardening in the plastic phase for simplicity

- The shear stress is uniform throughout the laminate and takes the value

$$\sqrt{3}\sigma_{12} = R_0 + R = R_0 + H_\chi(p - p_\chi) = R_0 - Ap_{\chi,11}$$

- Derivation of the previous equations with respect to x_1 shows that $p_{\chi,111} = 0$ which leads to the parabolic profile of the micro–plastic deformation in the soft phase

$$p_\chi(x) = \alpha x^2 + \beta, \quad \forall |x| \leq \frac{s}{2}$$

  Note that

$$\sqrt{3}\sigma_{12} = R_0 - 2A\alpha$$

- The parabolic plastic strain profile follows

$$p = \alpha x^2 + \beta - \frac{2A}{H_\chi}$$

Resolution of the b.v.p.

A new feature of the model is that the microplastic strain p_χ does not vanish in general in the hard phase, whereas p does:

$$p_\chi - \frac{A^h}{H_\chi}\Delta p_\chi = 0$$

$$p_\chi^h = \alpha_h \cosh \omega_h (x - \frac{l}{2}), \quad \frac{s}{2} \leq x \leq \frac{s}{2} + h, \quad \text{with} \quad \omega_h^2 = \frac{H_\chi}{A_h}$$

the p_χ^h profile is of hyperbolic nature

Interface conditions

- Continuity of micro–plastic deformation at $x = s/2$:

$$\alpha \frac{s^2}{4} + \beta = \alpha_h \cosh \omega_h \frac{h}{2}$$

- Continuity of the generalized stress component b_1:

$$A\alpha s = -A_h \alpha_h \omega_h \sinh \omega_h \frac{h}{2}$$

Interface conditions

The displacement in the plastic and elastic phases can be expressed as

$$u^s = \alpha \frac{x^3}{\sqrt{3}} + \left(\sqrt{3}\beta - \bar{\gamma} + \frac{R_0}{\sqrt{3}\mu} - 2A\alpha(\frac{1}{\sqrt{3}\mu} + \frac{\sqrt{3}}{H_\chi}) \right) x$$

$$u^h = \left(\frac{1}{\sqrt{3}\mu}(R_0 - 2A\alpha) - \bar{\gamma} \right) x + C$$

They are used to exploit two additional interface conditions

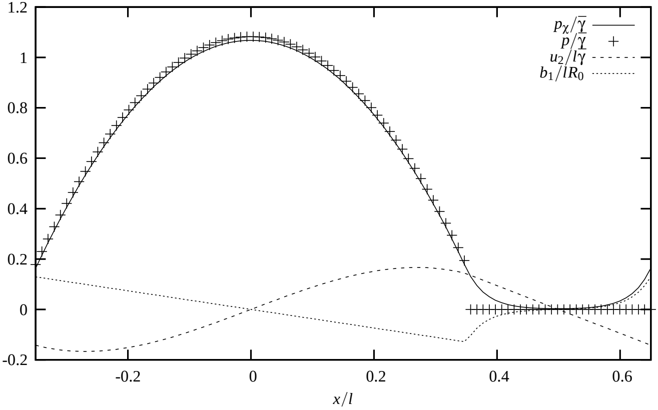
- Continuity of the displacement at $x = s/2$:
  $u^s(s/2) = u^h(s/2)$

$$\alpha \frac{s^3}{8\sqrt{3}\mu} + \sqrt{3}(\beta - \frac{2A\alpha}{H_\chi})\frac{s}{2} = C$$

- Periodicity of the displacement component
  $u^s(-s/2) = u^h(s/2 + h)$

$$-(\frac{\sigma_{12}}{\mu} - \bar{\gamma})l + \sqrt{3}(\beta + \frac{2A\alpha}{H_\chi})\frac{s}{2} - \alpha \frac{s^3}{8\sqrt{3}} = C$$

Plastic strain profiles in the channel



μ (MPa)	R_0 (MPa)	H_χ (MPa)	A (MPa.mm^2)	f	l (μm)	$\bar{\gamma}$
30000	20	50000	0.005	0.7	10	0.01

Overall size effect

The scaling law results from the expression of the overall stress σ_{12} as a function of the mean plastic strain over the unit cell:

$$\bar{p} = \frac{1}{l}\int_{-\frac{s}{2}}^{\frac{s}{2}}(\alpha x^2 + \beta - \frac{2A\alpha}{H_\chi})\,dx = \beta f\left(1 - \frac{1}{L^2}\left(\frac{s^2}{12} - \frac{2A}{H_\chi}\right)\right)$$

with $L^2 = \dfrac{s^2}{4} + \dfrac{A}{A_h}\dfrac{s}{\omega_h}\mathrm{cotanh}(\omega_h\dfrac{h}{2}) = -\dfrac{\beta}{\alpha}$. The uniform stress component can now be expressed as a function of the volume fraction f of the soft phase and of the unit cell size l:

$$\sqrt{3}\sigma_{12} = R_0 + \frac{2A}{f}\frac{\bar{p}}{\dfrac{f^2l^2}{6} + \dfrac{2A}{H_\chi} + \dfrac{A}{A_h}\dfrac{fl}{\omega_h}\mathrm{cotanh}\left(\omega_h\dfrac{h}{2}\right)}$$

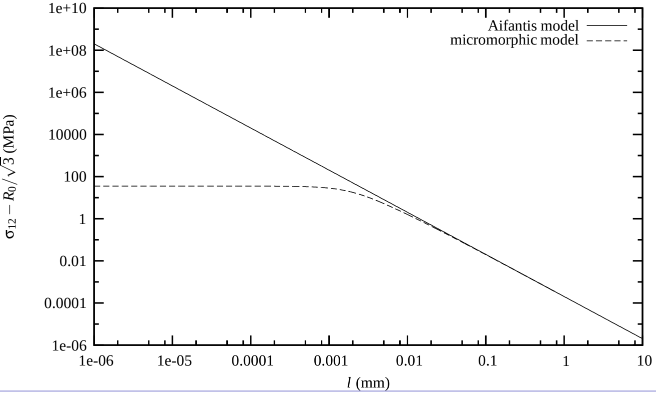
displaying a size–dependent overall linear hardening

Scaling laws

Two limit cases naturally arise

- Internal constraint $H_\chi \to \infty$ for which the strain gradient plasticity model is retrieved
- Unit cell size $l \to 0$ leads to saturation stress

$$\sqrt{3}\sigma_{12} - R_0 \sim H_\chi \frac{1-f}{f} \bar{p}$$



Plan

Main comments

- The choice of quadratic potentials with respect to strain gradients leads to the existence of a size dependent overall linear hardening modulus in a laminate microstructure
- The corresponding scaling law according to Aifantis strain gradient plasticity (one new material parameter) is $1/l^2$
- In contrast the micromorphic model (two new material parameters) leads to a saturation at nano–scales
- Physical metallurgy hints rather at scaling laws of the Orowan $(1/l)$ or Hall-Petch $(1/\sqrt{l})$ types

Needed improvements

- Improve constitutive equations; for instance

$$\rho\psi(\boldsymbol{\nabla} p) = \sqrt{\boldsymbol{\nabla} p \cdot \underset{\sim}{\mathbf{A}} \cdot \boldsymbol{\nabla} p}$$

  [Conti and Ortiz, 2005, Okumura et al., 2007]

- Add higher order dissipative parts
  [Forest, 2009, Anand et al., 2012]
- Enhance interface conditions
  [Gurtin and Needleman, 2005, Acharya, 2007, Gurtin and Anand, 2008]

Extension to crystal plasticity

- In crystal plasticity, the relevant variable is not the gradient of the cumulative plastic strain but rather the dislocation density tensor

$$\underset{\sim}{\boldsymbol{\Gamma}} = -\mathrm{curl}\, \underset{\sim}{\mathbf{H}}^p, \quad \underset{\sim}{\mathbf{H}} = \underline{\mathbf{u}} \otimes \boldsymbol{\nabla} = \underset{\sim}{\mathbf{H}}^e + \underset{\sim}{\mathbf{H}}^p$$

  [Cermelli and Gurtin, 2001, Svendsen, 2002]
  but no effects in laminates...
- A strain gradient and a micromorphic theory can be designed based on the introduction of the dislocation density tensor in the free energy function [Aslan et al., 2011]
- Similar effects arise in laminate microstructures but the overall linear hardening is of kinematic nature

$$x = \mathrm{curl}\,\mathrm{curl}\, \underset{\sim}{\mathbf{H}}^p : \underline{\mathbf{m}} \otimes \underline{\mathbf{n}}$$

  [Cordero et al., 2010, Cottura et al., 2012]
- Hall–Petch related size effects can be predicted in polycrystals based on full field simulations
  [Forest et al., 2000, Bayley et al., 2007, Neff et al., 2009, Bargmann et al., 2010, Cordero et al., 2012, Wulfinghoff and Boehlke, 2012]
- Constitutive equations are still unrealistic...

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