Friction Induced Vibrations: Oscillatory Instability with Dissipative and Gyroscopic Influences

On Modelling and Simulation of Brake-Squeal

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Kurzfassung

Bremsengeräusche – allen voran das Quietschen – stellen nach wie vor ein Problem dar. Trotz detaillierter Modellierung ist die Diskrepanz zwischen Simulation und experimentellen Ergebnissen mitunter groß, so dass seitens der Simulation offensichtlich grundsätzliche Zusammenhänge nach wie vor nicht verstanden sind. In diesem Beitrag wird die prinzipielle Form der Bewegungsgleichungen von Systemen bewegter elastischer Körper mit Reibung angegeben und gezeigt, dass Führungsbewegungen gyroskopische Anteile erzeugen. Hinsichtlich der Kontaktformulierung werden Lagrange-Multiplikatoren und ein Penalty-Verfahren gegenübergestellt. Die Linearisierung der Reibungskräfte führt hierbei zu weiteren gyroskopischen und darüber hinaus zu zirkulatorischen Einflüssen, welche oszillatorische Instabilität (Flutter) bewirken können. Es zeigt sich zudem, dass gyroskopisch-zirkulatorische Systeme ohne Dämpfung zwingend instabil sind. Der Modellierung von Dämpfung kommt somit eine besondere Rolle zu. Außerdem wird kurz auf die u.U. destabilisierende Wirkung schwacher Dämpfung eingegangen („Ziegler Paradoxon“).

Abstract

Despite all efforts in the past, brake squeal is still a problem. Even detailed simulations only hardly allow for good predictions and the discrepancy to experiments may be considerable from case to case. This paper revisits the general pattern of the equations of motion of moving continua and shows how transport motions provoke gyroscopic terms. Concerning the contact formulation, Lagrange-Multipliers and a penalty technique are compared. Furthermore, the linearization of the friction forces gives rise to gyroscopic and circulatory matrices, where the latter may cause flutter instability. Moreover, it is deduced that undamped gyroscopic-circulatory systems are always unstable. Hence, a thorough modelling of damping will be a key issue. Further, it is shown that small damping may have the surprising effect to destabilize a stable system (“Ziegler’s paradox”).

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1. Introduction

Although some problems in technical application may be modelled using rigid bodies, there are many problems which have to be addressed using flexible bodies.

One of the foremost examples of such problems are friction induced vibrations like brake-squeal, which require the consideration of the elastic properties of the contacting bodies as well as a thorough modelling of the contact. Today, this class of problems has been investigated for decades – however, simulation results still show significant differences to experiments from case to case. Hence, a deeper understanding of the peculiarities that come along with the simulation of friction induced vibrations is necessary.

In the following, the principle form of the equations of motion of such mechanical systems will be derived for the example of a rotating disc which is in contact with two brake pads. Despite the simplicity of this small system, all important effects will be observable. In a second step, the arising instability szenarios are discussed.

Although being rather general, the discussion will be orientated along standard tools and algorithms in structural dynamics with contacts.

2. Systems of moving continua

Equations of motion without contact contributions

As an example problem, a system comprising a disc-brake rotor and two pads will be examined (fig. 1). The dynamics of this example system of flexible bodies may be described using the principle of virtual work [1], [2]. In the following, $B^{(\alpha)}$ denotes the set of material points of a body α ($\alpha = \{D,1,2\}$), while the set of its surface points is refered to as $\partial B^{(\alpha)}$ (fig. 1). Here, the index D refers to the disc while the indices $1,2$ refer to the upper and lower pad.

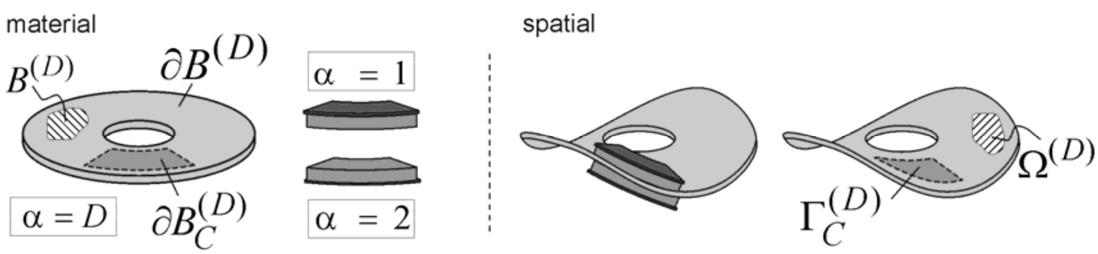


Fig. 1: Members of the example system: rotor, brake pads.

Assuming all system members to be linearly elastic bodies and the virtual work contributions only to originate from linear inner damping within body α and the contact contributions in the contact zone $\partial B_C^{(\alpha)}$, i.e. $\delta W^{(\alpha)} = \delta W_C^{(\alpha)} + \delta W_D^{(\alpha)}$, the principle of virtual work in material form reads

$$\sum_{\alpha} \int_{B^{(\alpha)}} \delta\vec{r}_\alpha \cdot \rho \vec{a}_\alpha + \delta\vec{\vec{\varepsilon}}_\alpha \cdot \left[\vec{\vec{\vec{\vec{D}}}} \dot{\vec{\vec{\varepsilon}}}_\alpha + \vec{\vec{\vec{\vec{K}}}} \vec{\vec{\varepsilon}}_\alpha \right] \mathrm{d}V = \sum_{\alpha} \int_{\partial B_C^{(\alpha)}} \delta\vec{r}_\alpha \cdot \vec{F}_C^{(\alpha)} \,\mathrm{d}A \quad , \tag{1}$$

$$\mathrm{N}_D^{(\alpha)}[\vec{r}_\alpha] = 0 \quad \text{on} \quad \partial B_D^\alpha \quad \text{(Dirichlet BC)} ,$$

where $\vec{r}_\alpha$, $\vec{v}_\alpha$, $\vec{a}_\alpha$ denote the position, velocity and acceleration vectors of a material point, $\vec{\vec{\varepsilon}}$ is the strain tensor and $\vec{\vec{\vec{\vec{D}}}}$ and $\vec{\vec{\vec{\vec{K}}}}$ are the dissipation tensor and the elasticity tensor of body α. Transport motions are assumed to be kinematically prescribed and hence driving forces do not appear in the virtual work balance.

Beyond other advantages, choosing this principle of analytical mechanics, instead of applying Newton’s Law, guarantees for symmetric mass and stiffness matrices after discretization. As will be shown later, the symmetry properties of the system matrices may be used in order to judge the stability behaviour. Finally, it has to be pointed out that the principle of virtual work is usually used as starting point for FEM-formulations: hence, all following considerations directly apply on the results of FEM schemes.

For concrete evaluation, the tensor-valued quantities have to be decomposed using some coordinate system. Assuming $\vec{\vec{\varepsilon}}$ to be symmetric, the voigt notation

$$\mathbf{e} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & \varepsilon_{23} & \varepsilon_{13} & \varepsilon_{12} \end{bmatrix}^{\mathrm{T}} \tag{2}$$

is used to rewrite the scalar products of dyadic tensors in equation (1) as matrix expressions:

$$\delta\vec{\vec{\varepsilon}} \cdot \left[\vec{\vec{\vec{\vec{D}}}} \dot{\vec{\vec{\varepsilon}}} + \vec{\vec{\vec{\vec{K}}}} \vec{\vec{\varepsilon}} \right] \quad \rightarrow \quad \delta\mathbf{e}^{\mathrm{T}} [\mathfrak{D}\dot{\mathbf{e}} + \mathfrak{K}\mathbf{e}] . \tag{3}$$

With this notation, index notation can be avoided and all statements can comfortably be done in terms of matrix algebra.

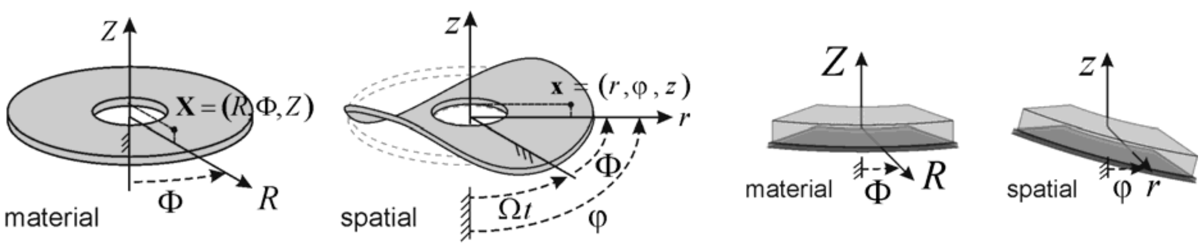


Fig. 2: Coordinate systems of the disc and a pad.

A particle of the disc may be addressed by material (Lagrange) coordinates $\mathbf{X} = (R, \Phi, Z)$. Then, its spatial position, decomposed in a spatial coordinate system, reads

$$\mathbf{r}_D(\mathbf{X},t) = \mathbf{b}_D + \mathbf{u}_D(\mathbf{X},t)\,, \qquad (4)$$

where $\mathbf{r}$ is the spatial position, $\mathbf{b}$ is a prescribed part due to the transport motion and $\mathbf{u}$ is the displacement field addressed by material coordinates. The corresponding velocity and acceleration of a fixed particle $\mathbf{X}$ in the inertial frame read

$$\mathbf{v}_D(\mathbf{X},\mathbf{t}) = \left.\frac{{}^I\mathrm{d}}{\mathrm{d}t}\right|_{\mathbf{X}} \mathbf{r}_D = \frac{{}^I\partial}{\partial t}\mathbf{r}_D = \dot{\mathbf{r}}_D(\mathbf{X},t) = \mathring{\mathbf{b}}_D + \mathring{\mathbf{u}}_D(\mathbf{X},\mathbf{t})\;,\; \mathbf{a}_D(\mathbf{X},\mathbf{t}) = \frac{{}^I\partial}{\partial t}\mathbf{v}_D = \mathring{\mathring{\mathbf{b}}}_D + \mathring{\mathring{\mathbf{u}}}_D(\mathbf{X},\mathbf{t}) \qquad (5)$$

where $\frac{{}^I\partial}{\partial t}(\;) = (\mathring{\;})$ indicates differentiation for a fixed particle ($\mathbf{X} = \mathrm{const}$) with respect to the inertial frame. However, it is still necessary to account for possible time-dependence of the used frame, when calculating the time derivative $(\mathring{\;})$ (e.g. if polar-coordinates are used). Since for a stationary motion the contact zones between the disc and the pads will be spatially invariant, it is often advantageous to describe the displacement field of the disc in terms of spatial Euler-coordinates $\mathbf{x} = (r,\varphi,z)$ instead of material Lagrange-coordinates, i.e. $\mathbf{u} = (\mathbf{x},t)$. Usually, a part of the velocity field may be interpreted as rigid body motion. With this notion, the angular velocity Ω about the z-axis may be introduced, linking spatial and material coordinates by

$$r = R \quad,\quad \varphi = \Omega t + \Phi \quad,\quad z = Z\,. \qquad (6)$$

Thus, $\mathbf{x} = \mathbf{x}(\mathbf{X},t)$ is a one-to-one mapping between the material coordinates (reference placement) and the spatial coordinates (to describe the current placement).
The time derivatives of the new field variable $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ with respect to the inertial frame read

$$\dot{\mathbf{u}}_D(\mathbf{x},t) = \frac{{}^\mathrm{I}\mathrm{d}}{\mathrm{d}t}\mathbf{u}_D = \frac{{}^\mathrm{I}\partial}{\partial t}\mathbf{u}_D + \frac{\partial}{\partial\varphi}\mathbf{u}_D\left(\frac{\mathrm{d}}{\mathrm{dt}}\varphi\right) = \frac{\partial}{\partial t}\mathbf{u}_D + \Omega\mathbf{r}'_D(\mathbf{x},t) \quad \text{with} \quad (\;)' = \frac{\partial}{\partial\varphi}\;,\; (\dot{\;}) = \frac{{}^I\mathrm{d}}{\mathrm{d}t}$$

$$\begin{aligned}\ddot{\mathbf{u}}_D(\mathbf{x},t) &= \frac{{}^\mathrm{I}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{u}_D = \frac{{}^\mathrm{I}\partial^2}{\partial t^2}\mathbf{u}_D + 2\frac{{}^\mathrm{I}\partial^2}{\partial\varphi\,\partial t}\mathbf{u}_D\left(\frac{\mathrm{d}}{\mathrm{dt}}\varphi\right) + \frac{\partial^2}{\partial\varphi^2}\mathbf{u}_D\left(\frac{\mathrm{d}}{\mathrm{dt}}\varphi\right)^2 \\ &= \frac{{}^\mathrm{I}\partial^2}{\partial t^2}\mathbf{u}_D(\mathbf{x},t) + 2\Omega\dot{\mathbf{u}}'_D(\mathbf{x},t) + \Omega^2\mathbf{u}''_D(\mathbf{x},t).\end{aligned} \qquad (7)$$

As before, the leading superscript I indicates that time derivatives have to be calculated with respect to the inertial frame and the partial derivative means differentiation for $\mathbf{X} = \mathrm{const}$. Noting that the time-derivatives in (6) are built for fixed $\mathbf{X}$, i.e. $\dot{\mathbf{u}}_D(\mathbf{X},t) = \frac{\partial}{\partial t}\mathbf{u}_D(\mathbf{X},t)$ and $\mathbf{x}$, $\mathbf{X}$ instantaneously refer to the same particle, i.e. $\dot{\mathbf{u}}_D(\mathbf{X},t) = \frac{\partial}{\partial t}\mathbf{u}_D(\mathbf{X}(\mathbf{x}),t) = \frac{\partial}{\partial t}\mathbf{u}_D(\mathbf{x},t)$, one finds the following notions in terms of spatial coordinates

$$\mathbf{r}_D = \mathbf{b}_D + \mathbf{u}_D(\mathbf{x},t) \quad , \quad \mathbf{v}_D = \dot{\mathbf{b}}_D + \dot{\mathbf{u}}_D(\mathbf{x},t) - \Omega \mathbf{u}'_D(\mathbf{x},t) \quad \text{and}$$
$$\mathbf{a}_D = \ddot{\mathbf{b}}_D + \ddot{\mathbf{u}}_D(\mathbf{x},t) - 2\Omega \dot{\mathbf{u}}'_D(\mathbf{x},t) - \Omega^2 \mathbf{u}''_D(\mathbf{x},t) \,. \qquad (8)$$

For small strains, the coordinate matrix $\boldsymbol{\varepsilon}_D$ of the strain tensor $\vec{\vec{\varepsilon}}_D$ can be determined from

$$\boldsymbol{\varepsilon}_D = \frac{1}{2}\left(\mathrm{Grad}\,\mathbf{u}_D(\mathbf{X},t) + \mathrm{Grad}^{\mathrm{T}}\,\mathbf{u}_D(\mathbf{X},t)\right) = \frac{1}{2}\left(\mathrm{grad}\,\mathbf{u}_D(\mathbf{x}\,,t) + \mathrm{grad}^{\mathrm{T}}\,\mathbf{u}_D(\mathbf{x}\,,t)\right) \qquad (9)$$

and finally resorting these coordinates yields the strain vector $\mathbf{e}$.

Analogously, particles of the brake pads can be addressed by their material coordinates $\mathbf{X}$ as well as by the according spatial coordinates $\mathbf{x} = \mathbf{x}(\mathbf{X},t)$. Since the pads are not subjected to any prescribed transport motion, material and spatial system read $\mathbf{x} = \mathbf{X}$. Thus the kinematical quantities of the pads (corresponding to (8)) simplify to

$$\mathbf{r}_{1/2} = \mathbf{X}(\mathbf{x}) + \mathbf{u}_{1/2}(\mathbf{x},t) \ , \ \mathbf{v}_{1/2} = \dot{\mathbf{u}}_{1/2}(\mathbf{x},t) \quad \text{and} \quad \mathbf{a}_{1/2} = \ddot{\mathbf{u}}_{1/2}(\mathbf{x},t) \,. \qquad (10)$$

Finally, the differentials and integration domains have to be rewritten to spatial coordinates. The differential volumes in material and spatial representation are connected by $\mathrm{d}v = J\,\mathrm{d}V$ where $J = \det(\mathbf{F})$ is the determinant of the deformation gradient $\mathbf{F} = \mathrm{grad}(\mathbf{u}) + \mathbf{1}$. For small strains, $\|\mathrm{grad}\,\mathbf{u}\| << 1$ and hence $\det(\mathbf{F}) \approx 1$. In a similar way, one finds $\mathrm{d}a \approx \mathrm{d}A$ for the surface differentials. Since in the sense of stability analyses stationary motions will be assumed, the contact domains are time invariant. Thus the corresponding contact domains in spatial coordinates read $\Gamma_C^{(\alpha)} = \left\{\mathbf{x}(\mathbf{X}) \,|\, \mathbf{X} \in \partial B_C^{(\alpha)}\right\}$.

The discretizations of the field variables in the partial differential equations arising from (1) can generally be done by a separation approach using spatial shape functions $\mathbf{\Phi}(\mathbf{x})$ and time-dependent amplitude functions $\mathbf{q}(t)$, i.e. $\mathbf{u}(\mathbf{x},t) \approx \mathbf{\Phi}^{\mathrm{T}}(\mathbf{x})\,\mathbf{q}(t)$. For a body α, this reads

$$\mathbf{u}_\alpha \approx \mathbf{\Phi}_\alpha^{\mathrm{T}}\mathbf{q}_\alpha \ , \ \ddot{\mathbf{u}}_\alpha \approx \mathbf{\Phi}_\alpha^{\mathrm{T}}\ddot{\mathbf{q}}_\alpha \ , \ \mathbf{u}'_\alpha \approx \mathbf{\Phi}_{\alpha,\varphi}^{\mathrm{T}}\mathbf{q}_\alpha \ , \ \dot{\mathbf{u}}'_\alpha \approx \mathbf{\Phi}_{\alpha,\varphi}^{\mathrm{T}}\dot{\mathbf{q}}_\alpha \ , \ \text{etc.} \qquad (11)$$

where $\alpha \in \{D,1,2\}$. The corresponding strain matrices $\mathbf{e}_D$ (cf. eq. (2)) may then be written as

$$\mathbf{e}_\alpha \approx \mathbf{B}_\alpha^{\mathrm{T}}\mathbf{q}_\alpha \ , \quad \dot{\mathbf{e}}_\alpha \approx \mathbf{B}_\alpha^{\mathrm{T}}\dot{\mathbf{q}}_\alpha \ , \qquad (12)$$

where the Matrix $\mathbf{B}_\alpha$ contains partial spatial derivatives of the shape functions. Discretizing the kinematic quantities (4)-(10) using these ansatzes, inserting them into (1) and carrying out the spatial volume and surface integrations eventually yields

$$\begin{bmatrix} \mathbf{M}_D & & \\ & \mathbf{M}_1 & \\ & & \mathbf{M}_2 \end{bmatrix}\begin{pmatrix} \ddot{\mathbf{q}}_D \\ \ddot{\mathbf{q}}_1 \\ \ddot{\mathbf{q}}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{D}_D + \Omega\mathbf{G} & & \\ & \mathbf{D}_1 & \\ & & \mathbf{D}_2 \end{bmatrix}\begin{pmatrix} \dot{\mathbf{q}}_D \\ \dot{\mathbf{q}}_1 \\ \dot{\mathbf{q}}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{K}_D - \Omega^2\mathbf{M}^* & & \\ & \mathbf{K}_1 & \\ & & \mathbf{K}_2 \end{bmatrix}\begin{pmatrix} \mathbf{q}_D \\ \mathbf{q}_1 \\ \mathbf{q}_2 \end{pmatrix} = \begin{bmatrix} \int\limits_{\Gamma_C} \mathbf{\Phi}_D \mathbf{f}_D^C \,\mathrm{d}a \\ \int\limits_{\Gamma_C} \mathbf{\Phi}_1 \mathbf{f}_1^C \,\mathrm{d}a \\ \int\limits_{\Gamma_C} \mathbf{\Phi}_2 \mathbf{f}_2^C \,\mathrm{d}a \end{bmatrix} \qquad \textbf{(13)}$$

with the sub-matrices

$$\mathbf{M}_\alpha = \int \rho_\alpha \mathbf{\Phi}_\alpha \mathbf{\Phi}_\alpha^{\mathrm{T}} \mathrm{d}v \quad , \quad \mathbf{G} = -\int \rho_\alpha \mathbf{\Phi}_\alpha \mathbf{\Phi}_{\alpha,\varphi}^{\mathrm{T}} \,\mathrm{d}v \qquad (\alpha = \{D,1,2\})$$

$$\mathbf{D}_\alpha = \int \mathbf{B}_\alpha \mathcal{D}_\alpha \mathbf{B}_\alpha^{\mathrm{T}} \mathrm{d}v \quad , \quad \mathbf{K}_\alpha = \int \mathbf{B}_\alpha \mathcal{K}_\alpha \mathbf{B}_\alpha^{\mathrm{T}} \mathrm{d}v \quad , \quad \mathbf{M}^* = \int \rho_\alpha \mathbf{\Phi}_\alpha \mathbf{\Phi}_{\alpha,\varphi\varphi}^{\mathrm{T}} \,\mathrm{d}v \,. \qquad \textbf{(14)}$$

By integration by parts, one can readily proof the following symmetry properties

$$\mathbf{M}_\alpha = \mathbf{M}_\alpha^{\mathrm{T}} \;,\; \mathbf{D}_\alpha = \mathbf{D}_\alpha^{\mathrm{T}} \;,\; \mathbf{G}_\alpha = -\mathbf{G}_\alpha^{\mathrm{T}} \;,\; \mathbf{K}_\alpha = \mathbf{K}_\alpha^{\mathrm{T}} \quad \text{and} \quad \mathbf{M}_\alpha^* = \mathbf{M}_\alpha^{*\,\mathrm{T}} . \qquad \textbf{(15)}$$

Finally, one comes up with equations of motion of the pattern

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{D} + \Omega\mathbf{G})\dot{\mathbf{q}} + (\mathbf{K} - \Omega^2\mathbf{M}^*)\mathbf{q} = \mathbf{F}$$

$$\text{where} \quad \mathbf{M} = \mathbf{M}^{\mathrm{T}} \,,\; \mathbf{D} = \mathbf{D}^{\mathrm{T}} \,,\; \mathbf{G} = -\mathbf{G}^{\mathrm{T}} \,,\; \mathbf{K} = \mathbf{K}^{\mathrm{T}} \,,\; \mathbf{M}^* = \mathbf{M}^{*\mathrm{T}} \,, \qquad \textbf{(16)}$$

which are typical for moving continua being described in spatial coordinates. It shall be emphasized that such equations will arise with analytical examinations using global ansatzes as well as with doing numerical analyses using commercial FEM packages.

Contact contributions

The forces on the right hand side of equation (16) arise from the discretization of the contact forces and can be split up in normal forces and tangential forces due to friction. Basically, the normal forces are constraint forces which prevent the interpenetration of the contacting bodies. Among others, there are two basic ways how to implement this constraint into the system description: the use of Lagrange multipliers or of a penalty formulation. In the following, it is assumed that there is only sliding friction, i.e. stiction will not occur.

Lagrange multipliers introduce the kinematic constraint via the work term

$$W_{C,N} = \int g(\mathbf{x})\lambda(\mathbf{x})\,\mathrm{d}a \quad \text{and its variation} \quad \delta W_{C,N} = \int \delta g(\mathbf{x})\lambda(\mathbf{x})\,\mathrm{d}a + \int g(\mathbf{x})\delta\lambda(\mathbf{x})\,\mathrm{d}a \qquad \textbf{(17)}$$

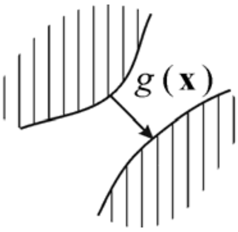
where the gap-function $g(\mathbf{x})$ describes the distance between both contact partners and the lagrange multiplier $\lambda(\mathbf{x})$ can be identified with the normal contact stress. With the ansatzes

$$g_{ij}(\mathbf{x}) = \mathbf{\theta}_i^{\mathrm{T}}\mathbf{q}_i - \mathbf{\theta}_j^{\mathrm{T}}\mathbf{q}_j \quad \text{and} \quad \lambda_{ij}(\mathbf{x}) = \mathbf{\Psi}_{ij}^{\mathrm{T}}\mathbf{\Lambda}_{ij} \qquad \textbf{(18)}$$

the contact between a body i and a body j can be discretized and after summation over all contact pairs $i-j$ of the system, some calculation eventually yields from (17).2

$$\delta W_{C,N} \approx \delta\mathbf{q}^{\mathrm{T}}\mathbf{N\Lambda} + \delta\mathbf{\Lambda}^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}\mathbf{q} \quad \text{where} \quad \mathbf{N} = \text{const}. \tag{19}$$

For Coulomb friction, the tangential traction vector reads $\boldsymbol{\tau}(\mathbf{x}) = \mu\lambda(\mathbf{x})\mathbf{e}_t$, where μ is the coefficient of sliding friction and $\mathbf{e}_t$ is the unit vector in the direction of the friction. In general, this direction can only be expressed using the local relative velocity as $\mathbf{e}_t = -\mathbf{v}_{rel}/\|\mathbf{v}_{rel}\|$ and hence in general $\mathbf{e}_t = \mathbf{e}_t(\mathbf{q},\dot{\mathbf{q}},\Omega)$ holds. Only in very special cases (e.g. 2d-models or holonomically guided contact points) it will be possible, to express the direction as a function of the generalized position coordinates, i.e. $\mathbf{e}_t = \mathbf{e}_t(\mathbf{q})$. In the following, it is assumed that the friction coefficient is constant and is the same in all contact areas ($\mu = \text{const}$).

a) 

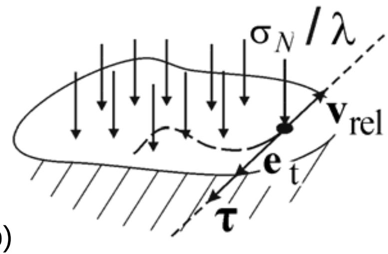
b) 

Fig. 3: Friction: a) Gap function. b) Normal tension and direction of the shear friction.

Discretization of $\lambda(\mathbf{x})$, carrying out the spatial surface integrals and summation over all contact pairs $i-j$, yields the total virtual work of the sliding friction as

$$\delta W_{C,T} \approx \mu\,\delta\mathbf{q}^{\mathrm{T}}\mathbf{T\Lambda} \quad \text{where} \quad \mu = \text{const}.\,, \quad \mathbf{T} = \mathbf{T}(\mathbf{q},\dot{\mathbf{q}},\Omega)\,, \tag{20}$$

where the parameter dependence of $\mathbf{T}$ results from $\mathbf{e}_t = \mathbf{e}_t(\mathbf{q},\dot{\mathbf{q}},\Omega)$.

Putting all together yields the equation of motion as differential-algebraic equation

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{q}} + (\mathbf{D} + \Omega\mathbf{G})\dot{\mathbf{q}} + \left(\mathbf{K} - \Omega^2\mathbf{M}^*\right)\mathbf{q} &= (\mathbf{N} + \mu\mathbf{T})\mathbf{\Lambda} \\ \mathbf{N}^{\mathrm{T}}\mathbf{q} &= \mathbf{0}\,. \end{aligned} \tag{21}$$

Solving for the vector of constraint forces gives

$$\begin{aligned} \mathbf{\Lambda} &= \left(\mathbf{N}^{\mathrm{T}}\mathbf{M}^{-1}(\mathbf{N} + \mu\mathbf{T})\right)^{-1}\left[\mathbf{N}^{\mathrm{T}}\mathbf{M}^{-1}(\mathbf{D} + \Omega\mathbf{G})\dot{\mathbf{q}} + \mathbf{N}^{\mathrm{T}}\mathbf{M}^{-1}\left(\mathbf{K} - \Omega^2\mathbf{M}^*\right)\mathbf{q}\right] \\ &= \mathbf{L}_0(\Omega,\mu,\mathbf{q},\dot{\mathbf{q}})\,\mathbf{q} + \mathbf{L}_1(\Omega,\mu,\mathbf{q},\dot{\mathbf{q}})\,\dot{\mathbf{q}}\,. \end{aligned} \tag{22}$$

Putting this into (21) and linearization about a stationary solution $\mathbf{q}_0$ with $\mathbf{q} = \mathbf{q}_0 + \Delta\mathbf{q}$ yields

$$\mathbf{M}\Delta\ddot{\mathbf{q}} + (\mathbf{D} + \Omega\mathbf{G} + \mathbf{P}_R(\Omega,\mu))\Delta\dot{\mathbf{q}} + \left(\mathbf{K} - \Omega^2\mathbf{M}^* + \mathbf{Q}_R(\Omega,\mu)\right)\Delta\mathbf{q} = \mathbf{0}\,, \tag{23}$$

which may be written as

$$\mathbf{M}\Delta\ddot{\mathbf{q}} + \mathbf{P}_L(\Omega,\mu)\Delta\dot{\mathbf{q}} + \mathbf{Q}_L(\Omega,\Omega^2,\mu)\Delta\mathbf{q} = \mathbf{0}\,. \tag{24}$$

Penalty-Formulations are based on the regularization of the constraint equation of non-interpenetration. Often, this is done by introducing the virtual work term

$$W_{C,N} = \frac{1}{2}\int k_C g(\mathbf{x})^2 \,\mathrm{d}a \quad \text{and its variation} \quad \delta W_{C,N} = \int k_C g(\mathbf{x})\,\delta g(\mathbf{x})\,\mathrm{d}a \qquad \textbf{(25)}$$

with the penalty parameter k_C. Beyond its meaning in the context of the contact formulation, it is usually interpreted in a physical sense as “contact stiffness”. Consequently, the normal tension $\sigma_N(\mathbf{x})$ in the contact is identified with $k_C g(\mathbf{x})$. Discretization of $g(\mathbf{x})$ eventually yields for general friction (i.e. $\mathbf{e}_t = \mathbf{e}_t(\mathbf{q},\dot{\mathbf{q}},\Omega)$) linearized perturbation equations of the form

$$\mathbf{M}\Delta\ddot{\mathbf{q}} + \left(\mathbf{D} + \Omega\mathbf{G} + k_C\widetilde{\mathbf{P}}_R(\Omega,\mu)\right)\Delta\dot{\mathbf{q}} + \left(\mathbf{K} - \Omega^2\mathbf{M}^* + k_C\widetilde{\mathbf{Q}}_R(\Omega,\mu)\right)\Delta\mathbf{q} = \mathbf{0}, \qquad \textbf{(26)}$$

which may be abbreviated as

$$\mathbf{M}\Delta\ddot{\mathbf{q}} + \mathbf{P}_P(k_C,\Omega,\mu)\Delta\dot{\mathbf{q}} + \mathbf{Q}_P(k_C,\Omega,\Omega^2,\mu)\Delta\mathbf{q} = \mathbf{0}. \qquad \textbf{(27)}$$

Here, it’s clear to see that the penalty parameter may have a strong influence on the system’s behaviour since it premultiplies all contributions arising from friction.

3. Eigenproblem

Small motions $\Delta\mathbf{q}$ of the system about a stationary solution $\mathbf{q}_0$ can be written in the form

$$\mathbf{M}\Delta\ddot{\mathbf{q}} + \left(\overline{\mathbf{D}} + \overline{\mathbf{G}}\right)\Delta\dot{\mathbf{q}} + \left(\overline{\mathbf{K}} + \mathbf{N}\right)\Delta\mathbf{q} = \mathbf{0}, \qquad \textbf{(28)}$$

where the symmetric and skewsymmetric parts of the velocity proportional terms have been collected in $\overline{\mathbf{D}} = \overline{\mathbf{D}}^{\mathrm{T}}$ and $\overline{\mathbf{G}} = -\overline{\mathbf{G}}^{\mathrm{T}}$, while $\overline{\mathbf{K}} = \overline{\mathbf{K}}^{\mathrm{T}}$ and $\mathbf{N} = -\mathbf{N}^{\mathrm{T}}$ carry the corresponding contributions due to positional forces. Please note, that all these matrices depend on the problem parameters μ, Ω and potentially on k_C.
The exponential ansatz $\Delta\mathbf{q} = \mathbf{u}\mathrm{e}^{\lambda t}$ produces the general quadratic eigenvalue problem

$$\left[\mathbf{M}\lambda^2 + \left(\overline{\mathbf{D}} + \overline{\mathbf{G}}\right)\lambda + \left(\overline{\mathbf{K}} + \mathbf{N}\right)\right]\mathbf{u} = \mathbf{0}. \qquad \textbf{(29)}$$

Left-multiplication by the conjugate-complex transpose $\overline{\mathbf{u}}^{\mathrm{T}}$ of the eigenvector $\mathbf{u}$ gives rise to conjugate-complex quadratic forms of the shape $\mathcal{R}(\mathbf{u}) = \overline{\mathbf{u}}^{\mathrm{T}}\mathbf{A}\mathbf{u}$.

At this point, the symmetry properties of the system matrices may be of great use. It holds for

- symmetric matrices $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$: $\mathcal{R} = \overline{\mathbf{u}}^{\mathrm{T}}\mathbf{A}\mathbf{u} \in \mathbb{R}$ for any $\mathbf{u} \in \mathbb{C}^{n\,x\,1}$ and $\mathcal{R} > 0$ for $\mathbf{A} > 0$
- skewsymm. matrices $\mathbf{B} = -\mathbf{B}^{\mathrm{T}}$:
  - $\mathcal{R} = \overline{\mathbf{u}}^{\mathrm{T}}\mathbf{B}\mathbf{u} = 0$ for any $\mathbf{u} \in \mathbb{R}^{n\,x\,1}$ and for any complex multiple $\mathbf{v} = \alpha\mathbf{u}$ ($\alpha \in \mathbb{C}$, $\mathbf{v} \in \mathbb{C}^{n\,x\,1}$)
  - $\mathcal{R} = \overline{\mathbf{u}}^{\mathrm{T}}\mathbf{B}\mathbf{u} = j\,b$, $b \in \mathbb{R}$, for any $\mathbf{u} \in \mathbb{C}^{n\,x\,1}$ which cannot be turned into a real quantity by a scalar complex factor α.

Generally, an arbitrary eigenvalue problem like (29) has complex valued eigenvectors $\mathbf{u} \in \mathbb{C}^{n \times 1}$, which cannot be written in a real-valued representation. It can be shown that for an undamped, nongyroscopic system $\left[\mathbf{M}\lambda^2 + \left(\overline{\mathbf{K}} + \mathbf{N}\right)\right]\mathbf{u} = 0$, which is marginally stable (i.e. $\Re(\lambda) = 0$), one can always find a set of entirely real valued eigenvectors. Furthermore, it is well known that these eigenvalues will also be eigenvalues of (29) iff $\mathbf{M}^{-1}\left(\overline{\mathbf{K}} + \mathbf{N}\right)$ and $\mathbf{M}^{-1}\left(\overline{\mathbf{D}} + \overline{\mathbf{G}}\right)$ commute, i.e. if $\left[\mathbf{M}^{-1}\left(\overline{\mathbf{D}} + \overline{\mathbf{G}}\right)\right]\left[\mathbf{M}^{-1}\left(\overline{\mathbf{K}} + \mathbf{N}\right)\right] = \left[\mathbf{M}^{-1}\left(\overline{\mathbf{K}} + \mathbf{N}\right)\right]\left[\mathbf{M}^{-1}\left(\overline{\mathbf{D}} + \overline{\mathbf{G}}\right)\right]$ holds. Only in this case (29) will have a full set of real eigenvectors.

With $m = \overline{\mathbf{u}}^{\mathrm{T}}\mathbf{M}\mathbf{u} \overset{!}{=} 1$ (mass normalized), $d = \overline{\mathbf{u}}^{\mathrm{T}}\mathbf{D}\mathbf{u}$, $k = \overline{\mathbf{u}}^{\mathrm{T}}\mathbf{K}\mathbf{u}$, $d = \overline{\mathbf{u}}^{\mathrm{T}}\mathbf{D}\mathbf{u}$ $jg = \overline{\mathbf{u}}^{\mathrm{T}}\mathbf{G}\mathbf{u}$, $jn = \overline{\mathbf{u}}^{\mathrm{T}}\mathbf{N}\mathbf{u}$ (29) yields

$$\lambda^2 + (d + jg)\lambda + (k + jn) = 0\,, \tag{30}$$

which is a quadratic equation for the eigenvalues

$$\lambda^{\pm} = -\frac{d + jg}{2} \pm \frac{1}{2}\sqrt{-4k - g^2 + d^2 + j(2dg - 4n)}\;. \tag{31}$$

Evaluation of the square-root is easily done by recasting the radicand in exponential form as

$$z = a + jb = r\,\mathrm{e}^{j2\varphi} \quad , r = \sqrt{a^2 + b^2} \quad , 2\varphi = \arg(z) \qquad \rightarrow \qquad \sqrt{z}^{\,\pm} = \pm\sqrt{r}\,\mathrm{e}^{j\varphi}\,. \tag{32}$$

Hence, the square root of a complex number produces two phasors (vectors in the complex plane), which have the length $\sqrt{r}$ and the angles φ and $\varphi + \pi$ to the real axis (cf. figure 4).

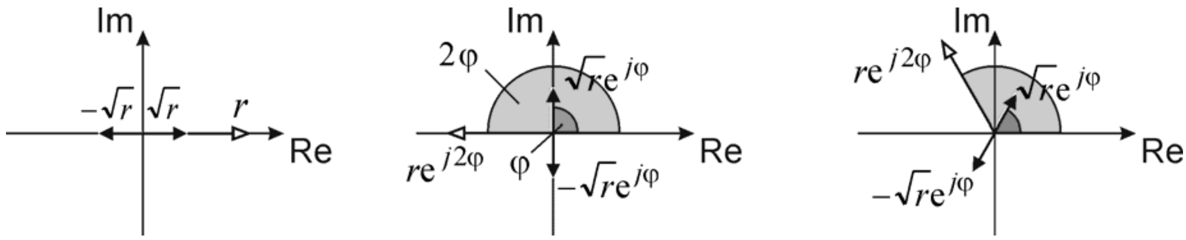


Fig. 4: Illustration of square roots of complex numbers.

Sometimes, the following notion of orthogonality of eigenvectors is used: using real quadratic forms, two eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$ are called orthogonal with respect to a matrix $\mathbf{A}$ if $\mathbf{u}_1^{\mathrm{T}}\mathbf{A}\mathbf{u}_2 = \mathbf{0}$ holds. For the undamped, non-gyroscopic system one can show that

$$(\lambda_1 - \lambda_2)\mathbf{u}_1^{\mathrm{T}}\mathbf{M}\mathbf{u}_2 = -2\mathbf{u}_1^{\mathrm{T}}\mathbf{N}\mathbf{u}_2 \tag{33}$$

holds for the eigenpairs $\{\lambda_1, \mathbf{u}_1\}$, $\{\lambda_2, \mathbf{u}_2\}$. Obviously, for different eigenvalues the corresponding eigenvectors are only orthogonal with respect to $\mathbf{M}$ if $\mathbf{N}$ vanishes or in particular cases if one of the vectors lies in the nullspace of $\mathbf{N}$.

4. Stability

Despite its simple structure, equation (29) does not allow for an easy assessment of the real part of the system’s eigenvalues – and hence the stability of the system. Introducing $\lambda = \rho + j\omega$ into equation (30), separation of real and imaginary part and finally applying the Routh-Hurwitz-criterion yields the necessary and sufficient conditions for asymptotic stability

$$d > 0 \quad \wedge \quad kd^2 + ndg - n^2 > 0 \quad . \qquad \textbf{(34)}$$

For $\mathbf{D} = \mathbf{0}$ or $\mathbf{u} \in \ker(\mathbf{D})$, thus $d = 0$, the system (or at least the mode corresponding to $\mathbf{u}$) can only be marginally stable, which cannot be checked by the Routh-Hurwitz-criterion. However, $\lambda = \rho + j\omega$ yields the relations

$$\rho^2 - \omega^2 - g\omega + k = 0 \quad , \quad \rho(2\omega + g) + n = 0\,, \qquad \textbf{(35)}$$

which has to be discussed from case to case. For further details see [3], [4], [6] for instance.

Noncirculatory systems

The simplest example is the undamped, nongyroscopic system $\left[\mathbf{M}\lambda^2 + \mathbf{K}\right]\mathbf{u} = 0$. The eigenvalues read $\lambda^{\pm} = \pm\sqrt{-k}$, which are purely imaginary numbers if $\mathbf{K}$ is positive definite and consequently $k > 0$ holds. Such systems are marginally stable and hence there is a complete set of real valued eigenvectors.

Following the theorem of Thomson-Tait-Chetayev, adding damping $\mathbf{D}$ and/or gyroscopic terms $\mathbf{G}$ will not destabilize the system, as long as $\mathbf{D}$ is at least positive semidefinit. If $d = \overline{\mathbf{u}}^{\mathrm{T}}\mathbf{D}\mathbf{u} > 0$ holds for all eigenvectors $\mathbf{u}$, the system is asymptotically stable: this is the case if $\mathbf{D} > 0$ holds or if the damping is pervasive, i.e. $d = \overline{\mathbf{u}}^{\mathrm{T}}\mathbf{D}\mathbf{u} > 0$ for all eigenvectors $\mathbf{u}$ even though $\mathbf{D} \geq 0$. Otherwise, motions may occur which are only marginally stable.

The stability of systems with vanishing circulatory terms and positive definite stiffness $\mathbf{K}$ is entirely determined by the damping $\mathbf{D}$. The root loci of damped systems with and without gyroscopic contributions are outlined in figure 5.

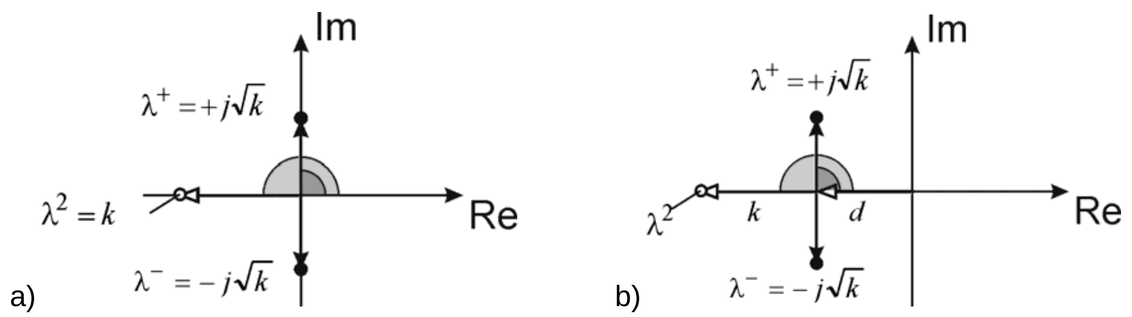


Fig. 5: Root loci of noncirculatory systems: a) undamped b) damped.

Circulatory Systems

Arbitrary systems which possess a circulatory matrix $\mathbf{N}$ may exhibit further instability szenarios, as will be demonstrated in the following.

Flutter instability may be seen as the standard paradigm of instability due to circulatory contributions. For example, such systems describe elastic systems with sliding friction, if transport motions (i.e. $\Omega \approx 0$ for disc-brake problems) as well as the velocity dependence of the friction tractions are neglected during linearization. Assume a system

$$\left[\mathbf{M}\lambda^2 + \left(\overline{\mathbf{K}} + p\overline{\mathbf{N}}\right)\right]\mathbf{u} = 0 \,, \qquad \textbf{(36)}$$

where the load parameter p controls the influence of the circulatory effects. It can be shown that if λ^2 is a solution of (36) then $\overline{\lambda}^2$ will be a solution as well. According to (33) the mutual mass-orthogonality of the set of eigenvectors $\mathbf{u}$ will be lost as p is changed from 0. As p further changes, some of the affected eigenvectors will mutually approach, i.e. $\mathbf{u}_r \to \mathbf{u}_s$ and in the limit $\mathbf{u}_r^T \mathbf{N}\mathbf{u}_s \to 0$ since $\mathbf{N}\mathbf{u}_s \perp \mathbf{u}_s$ and hence the corresponding eigenvalues approach as well (since $\mathbf{M} > 0$). The point, where two eigenvalues merge to a double root and the two corresponding eigenvectors align, is referred to as critical point and the corresponding parameter p_{crit} is denoted as critical load. By means of a vector series [1], [2] in the vicinity of this critical point, one can show that as p passes through p_{crit}, the eigenvalues $\lambda_r^{\pm} = \pm j\omega_r$, $\lambda_s^{\pm} = \pm j\omega_s$ change from purely imaginary roots to complex values off the imaginary axes. Accordingly, the eigenvectors $\mathbf{u}_r$, $\mathbf{u}_s$ change from real quantities to complex ones and hence, as p crosses p_{crit} the contribution n will no longer vanish in (30),(31). For $p > p_{\text{crit}}$, λ^2 leaves the real axis and the symmetry $\overline{\lambda}_r^2 = \lambda_s^2$ results in symmetric Rayleigh quotients according to $k_r = \overline{\mathbf{u}}_r^{\mathrm{T}} \mathbf{K}\mathbf{u}_r = k_s = k_{rs}$ and $n_r = \overline{\mathbf{u}}_r^{\mathrm{T}} \mathbf{N}\mathbf{u}_r = -n_s = n_{rs}$. Hence the behaviour of the eigenvalues may be summarized as

$$p < p_{\text{crit}}: \quad \begin{array}{l} \lambda_r^{\pm} = \pm\sqrt{-k_r} \\ \lambda_s^{\pm} = \pm\sqrt{-k_s} \end{array} \quad \xleftrightarrow[\text{critical point}]{p = p_{\text{crit}C}} \quad p > p_{\text{crit}}: \quad \begin{array}{l} \lambda_r^{\pm} = \pm\sqrt{-k_{rs} + jn_{rs}} \\ \lambda_s^{\pm} = \pm\sqrt{-k_{rs} - jn_{rs}} \end{array} \,. \qquad \textbf{(37)}$$

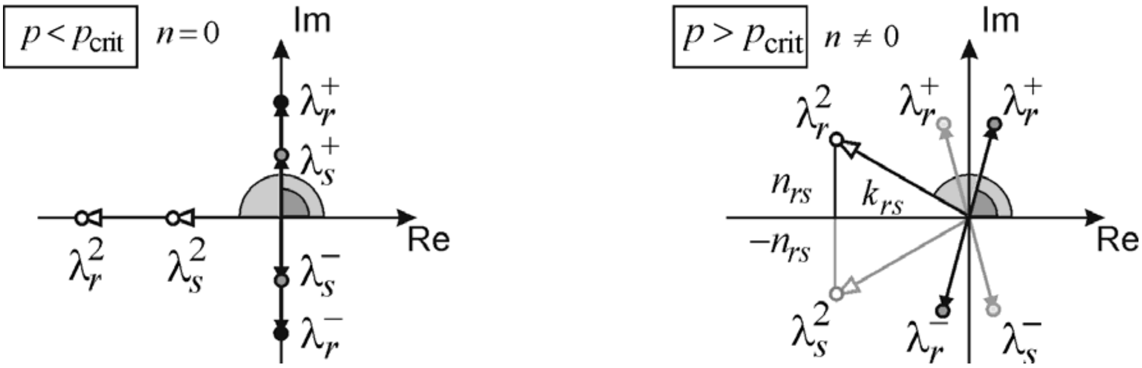


Fig. 6: Root loci of an undamped circulatory system: standard transition szenario to flutter.

The root loci during this transition are outlined in figure 6 and 7: as p passes through p_{crit}, the eigenvalues leave the imaginary axis to opposite directions and the system's stability changes from marginal stability to oscillatory instability (flutter). From figure 6 one may deduce graphically that $n \neq 0$ inevitably causes instability of the system. Additionally, this can be proofed analytically using equation (35).2, which states that $2\rho\omega + n = 0$ holds and hence two cases can be distinguished. For real valued eigenvectors, $n = 0$ holds and for $\omega \neq 0$ follows $\rho = 0$. However, for complex valued eigenvectors, $n \neq 0$ and hence $\rho \neq 0$ for $\omega \neq 0$.

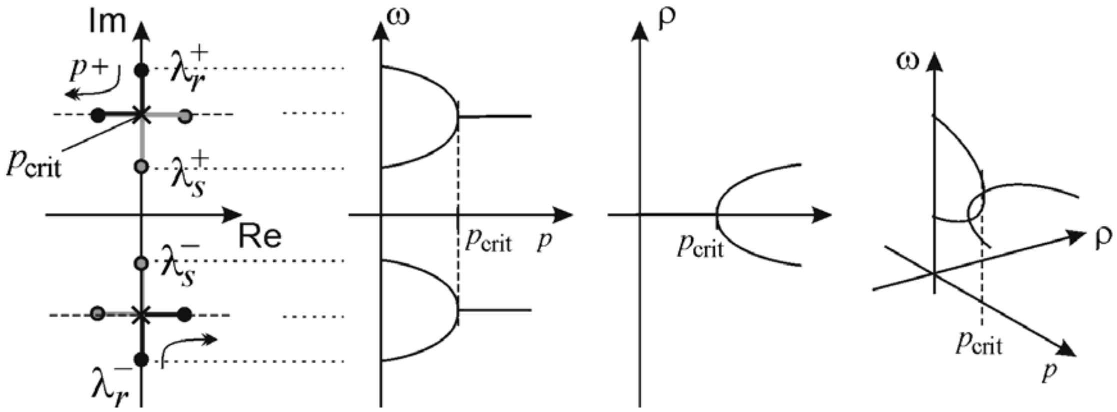


Fig. 7: Undamped circulatory system: path of root loci of $\lambda = \rho + j\omega$ as p passes p_{crit}.

Modal velocity proportional forces: adding velocity proportional terms $(\mathbf{D}+\mathbf{G})\dot{\mathbf{q}}$ to a circulatory system like (36), which fulfil the commutativity-condition, does not alter the eigenvectors of the underlying system. Hence, the transition of eigenvectors from real to complex ones will happen at the same critical parameter p_{crit}. Thus the eigenvalues are

$$\begin{aligned} p < p_{\text{crit}} \rightarrow \mathbf{u} \in \mathbb{R}^{n\times 1}: \quad & \lambda^{\pm} = -\frac{d}{2} \pm \frac{1}{2}\sqrt{-4k + d^2} \\ p > p_{\text{crit}} \rightarrow \mathbf{u} \in \mathbb{C}^{n\times 1}: \quad & \lambda^{\pm} = -\frac{d + jg}{2} \pm \frac{1}{2}\sqrt{-4k - g^2 + d^2 + j(2dg - 4n)} \end{aligned} \quad . \qquad \textbf{(38)}$$

As for system (36), the eigenvectors degenerate to one single eigenvector at $p = p_{\mathrm{crit}}$ and the considered system will have a single eigenvector to a double root as well.

Since real valued eigenvectors exist, the contributions jg, jn due to skew-symmetric matrices vanish and the stability condition (34) reads $d > 0 \;\wedge\; kd^2 > 0$. Hence, for $\mathbf{K} > 0$ the stability of such systems is entirely determined by the definiteness of $\mathbf{D}$. Furthermore, figure 8 shows that the stability border p_{stab} is no longer at the critical point, but instead $p_{\mathrm{stab}} > p_{\mathrm{crit}}$ holds.

A closer look at equation (23) or (26) reveals that the matrices $\mathbf{P}_L(\Omega, \Omega^2, \mu)$, $\mathbf{P}_P(k_C, \Omega, \Omega^2, \mu)$ of the velocity proportional forces depend on the system parameters like Ω, μ or the penalty parameter k_C. Thus, in general the commutativity conditions $(\mathbf{M}^{-1}\mathbf{P}_L)(\mathbf{M}^{-1}\mathbf{Q}) = (\mathbf{M}^{-1}\mathbf{Q})(\mathbf{M}^{-1}\mathbf{P}_L)$ or $(\mathbf{M}^{-1}\mathbf{P}_P)(\mathbf{M}^{-1}\tilde{\mathbf{Q}}) = (\mathbf{M}^{-1}\tilde{\mathbf{Q}})(\mathbf{M}^{-1}\mathbf{P}_P)$ are only fulfilled in very special cases for very particular sets of parameters Ω, μ and k_C. Hence, usually the velocity terms do not fulfil modality conditions and consequently systems in general will have complex valued eigenvectors.

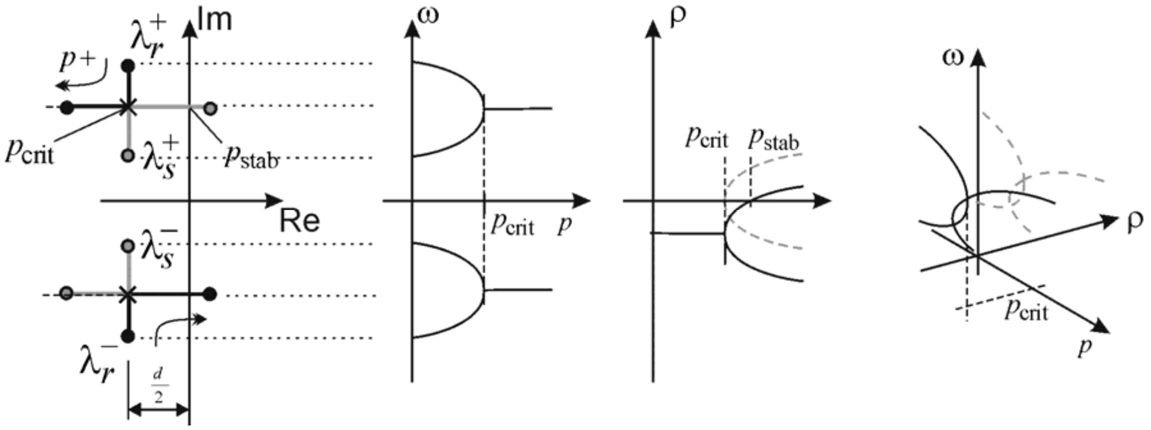


Fig. 8: Circulatory systems with modal velocity-proportional terms: path of root loci.

General damping and gyroscopic forces terms will cause complex valued eigenvectors. By means of series expansion, it is possible to show that adding non-modal velocity terms to a system with vanishing or only modal velocity terms, produces root loci that do not cross through a common critical point anymore (see figure 9). Further, the conditions for asymptotic stability now take the general form

$$d > 0 \;\wedge\; kd^2 + ndg - n^2 > 0 \quad . \qquad \textbf{(39)}$$

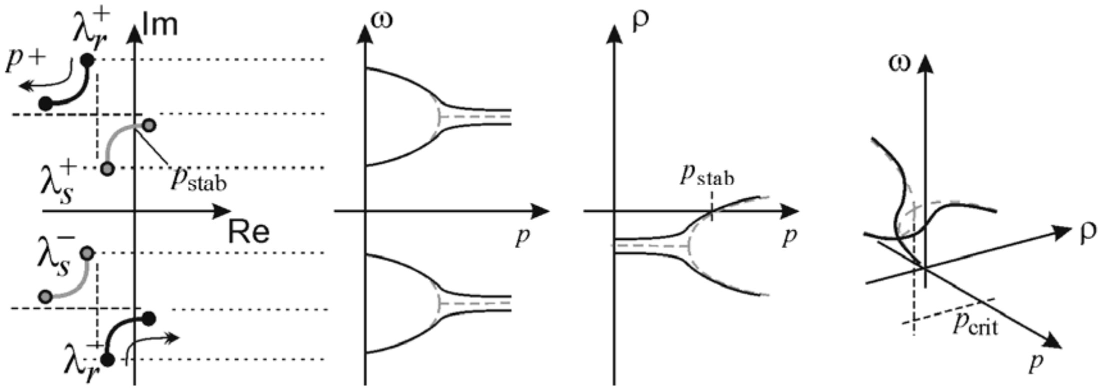


Fig. 9: Circulatory systems with non-modal velocity-proportional terms (general case).

For such systems, two interesting effects can be deduced:

1) Destabilization due to small damping: assume a marginally stable circulatory system $\left[\mathbf{M}\lambda^2 + \left(\overline{\mathbf{K}} + p\overline{\mathbf{N}}\right)\right]\mathbf{u} = 0$, which is near its flutter instability border (i.e. n is not small). Now, adding a small portion $\varepsilon(d + jg)$ ($\varepsilon << 1$) of positive definite or at least pervasive ($d > 0$) but non-modal velocity proportional terms turns the real eigenvectors to complex ones and hence yields from (39)

$$\varepsilon d > 0 \quad \wedge \quad -n^2 + \varepsilon^2\left(kd^2 + ndg\right) > 0 \quad . \tag{40}$$

While (40).1 is fulfilled, condition (40).2 will surely be violated since near the flutter border n will be much bigger than $O(\varepsilon^2)$. This surprising and rather contra-intuitive effect is often referred to as *Ziegler's Paradox* and has been extensively studied [5], [6].

2) Instability of undamped circulatory systems with gyroscopic contribution: assume a system of the form $\left[\mathbf{M}\lambda^2 + \overline{\mathbf{G}}\lambda + \left(\overline{\mathbf{K}} + p\overline{\mathbf{N}}\right)\right]\mathbf{u} = 0$ and recall that (almost) all eigenvectors are necessarily complex. Hence, for such eigenvectors asymptotic stability ($\rho < 0$) is ruled out by condition (34) and (35) shows that for $n \neq 0$ $\rho \neq 0$ must hold, thus the system is unstable. Although individual eigenvectors $\mathbf{u}$ in the nullspace of $\mathbf{G}$ can be real valued, thus allowing for marginal stability of individual vibration modes, the system is unstable since for $\mathbf{G} \neq \mathbf{0}$ there will be at least one eigendirection with $\rho > 0$. A rather academic case would be eigenvectors that either belong to the nullspace of $\mathbf{G}$ or that of $\mathbf{N}$, allowing for an entirely marginally stable system.

With equation (31) and the geometric interpretation of complex roots, one may easily illustrate this instability szenario (cf. fig. 10). Hence, such systems are (almost sure) unstable. An alternative explanation basing on the characteristic polynomial is given in [7].

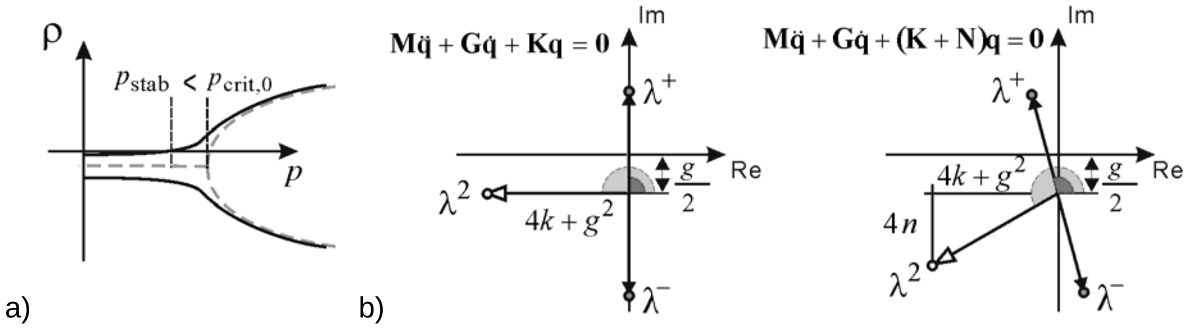


Fig. 10: Effects in circulatory systems with velocity dependent forces:
a) Ziegler's Paradox b) Instability of undamped gyroscopic circulatory systems.

5. Analysis of practical problems

Above findings shall be summarized in regard to the examination of practical problems. Although being generally valid, the following considerations will be accompanied exemplarily by the according commands in the software package `Abaqus®`. Starting with the simplest modelling approach, small vibrations of an elastic system without any contacts or driving motion, are described by

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0} \qquad \text{(assume } \mathbf{M} > \mathbf{0},\ \mathbf{K} > \mathbf{0}\text{)}. \qquad \textbf{(41)}$$

Superposing the transport motions (`*MOTION`) and accounting for the gyroscopic contributions (`*CORIO`) yields the gyroscopic system

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{G}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}\,. \qquad \textbf{(42)}$$

`*MOTION`
`*CORIO`

Adding friction (`*FRICTION`) to the system, linearization and calculation of eigenvalues (`*FREQUENCY , *COMPLEXFREQUENCY`) yields

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{G}\dot{\mathbf{q}} + (\mathbf{K} + \mathbf{K}_R + \mathbf{N}_R)\mathbf{q} = \mathbf{0}\,. \quad ⚡ \qquad \textbf{(43)}$$

`*FRICTION`
`*COMPLEXFREQUENCY`

As stated above, the trivial solution of this equation is inevitably unstable (indicated by the small lightning symbol). Hence, the influence of $\mathbf{G}$ (i.e. of the transport motion) may be critical in stability analyses. Moreover, positive-definiteness of $(\mathbf{K} + \mathbf{K}_R)$ is not sure and may be lost (e.g. if self-locking occurs) causing instability by divergence – but since flutter is commonly considered as the reason for squeal, $(\mathbf{K} + \mathbf{K}_R) > 0$ is assumed. Increasing the fidelity of the simulation by adding the velocity-proportional parts arising from the linearization of the friction as well (`*COMPLEXFREQUENCY FRICTIONDAMPING=YES`) produces

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{D}_R + \mathbf{G}_R + \mathbf{G})\dot{\mathbf{q}} + (\mathbf{K} + \mathbf{K}_R + \mathbf{N}_R)\mathbf{q} = \mathbf{0}\,, \quad ⚡ \qquad \textbf{(44)}$$

`FRICTIONDAMPING=YES`

which still represents a system whose trivial solution does not need to be stable, since the definiteness of $\mathbf{D}_R$ (i.e. the sign of d_R) in general is unknown. The final step to give a complete model is adding the damping $\mathbf{D}$ of the structure (`*DAMPING` for instance)

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{D} + \mathbf{D}_R + \mathbf{G}_R + \mathbf{G})\dot{\mathbf{q}} + (\mathbf{K} + \mathbf{K}_R + \mathbf{N}_R)\mathbf{q} = \mathbf{0}. \qquad \texttt{*DAMPING} \qquad \textbf{(45)}$$

Only this modelling stage allows for a reasonable stability analysis, since all steps before were per se unstable or at least uncertain. Apparently, a reliable assessment of circulatory systems intrinsically demands for considering transport motion as well as structural damping.

7. Conclusion

Describing moving continua – as the rotor of a disc-brake for instance – in terms of spatial coordinates always results in gyroscopic contributions to the linearized equation of motion about the trivial solution. Linearization of the friction forces yields further contributions to the system matrices. In particular, this linearization gives rise to non-symmetric parts in the positional matrix, which may provoke oscillatory instability (flutter), as well as it may produce general nonsymmetric contributions to the velocity dependent forces.

It is shown that the velocity-proportional forces on the system may have considerable influence on the stability behaviour. On the one hand, modelling gyroscopic circulatory systems without damping inevitably yields an unstable model – on the other hand, modelling of structural damping is a very difficult issue. Moreover, the importance of a realistic damping model is further emphasized by the effects of Ziegler’s Paradox, stating that small amounts of damping may cause a near-flutter system to be destabilized.

To sum up, it can be stated that increasing the fidelity of the simulations will demand for thorough modelling of the velocity dependent terms, i.e. damping and gyroscopic effects.

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