A multi-scale approach to plasticity: the *Continuum Dislocation Dynamics* theory

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GAMM summer school *Multiscale Material Modeling*,
Bad Herrenalb, 2012

*) in collaboration with: Michael Zaiser (University of Erlangen-Nürnberg), Thomas Hochrainer (University of Bremen) and Peter Gumbsch (Fraunhofer Institute IWM)

Part 0: Introduction

Part I: Overview small-scale plasticity

Part II: The *Continuum Dislocation Dynamics* theory
1. Theoretical foundations
2. Numerical examples and validation
3. Outlook

Part III: Introducing... the DFG Forschergruppe ‘Dislocation-based Plasticity’
Motivation: is COPPER = copper?

✓ centimeter-sized specimen can be used to predict meter-sized components

✗ Material behavior in small dimensions is not scale-invariant anymore

● How to predict plastic and hardening behaviour of components and devices?
● Scale of interest: several µm sizes becomes more and more important
● Influence of dislocations not negligible for:
  size effects, stochastic effects, physical hardening models

higher accuracy, less assumptions  computational efficiency
Stefan Sandfeld:
*A multiscale approach to plasticity: the Continuum Dislocation Dynamics theory,
GAMM summer school ‘Multiscale Material Modeling’, Bad Herrenalb, Germany, 2012*

- How to predict plastic and hardening behaviour of components and devices?
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The Continuum Dislocation Dynamics Theory (CDD) provides higher accuracy, less assumptions and computational efficiency compared to MD and DDD approaches.

- **MD**: movement of discrete lines
- **DDD**: internal stresses → analytical expressions
- **CDD**: evolution of dislocation density/curvature
  - internal stresses → statistics
  - computational cost independent of number of lines (density!)

**DDD**
- discrete dislocation dynamics
  - movement of discrete lines
  - internal stresses → analytical expressions
  - limit: number of (interacting) lines

**CDD**
- continuum dislocation dynamics
  - evolution of dislocation density/curvature
  - internal stresses → statistics
  - computational cost independent of number of lines (density!)

Phenomenological C.T.s
1. Macroscopic system
- boundaries and boundary conditions (BCs)
- external load: strain rate etc

2. Project stresses to slip planes from...
- external load, BCs (from 1)
- plastic strain (from 5)

3. Constitutive equations
- statistical model of microscopic stresses (dislocation interaction)
- dislocation velocity
\[ \tau = M : \sigma \]
\[ v \propto \tau_i(x) \]

4. Continuum Dislocation Dynamics (CDD)
- Evolve dislocation microstructure \( (\alpha^q) \)
- Time integration \( \rightarrow \) plastic slip \( \gamma \)

5. Homogenization
plastic slip \( \gamma \) on glide planes + compatibility condition
\( \rightarrow \) continuous plastic strain \( \varepsilon^{pl} \)
(macroscopic crystal)

For the next derivations and benchmark tests:
• we assume the dislocation velocity to be given
• no short range/long range stresses and interactions from dislocation microstructure
• this is unrealistic but makes sense as validation for the dislocation kinematics (=movement of lines or density)
The classical continuum theory of dislocations (Kröner, Nye, Bilby, Kondo in the ~1950s)

KRÖNER-NYE dislocation density tensor $\alpha$:

- $\alpha = \text{curl} \beta^\text{pl}$: inhomogeneous plastic distortion causes a dislocation density
- $\text{div} \alpha = 0$: dislocation lines do not start or end inside the crystal
- $\partial_t \alpha = \text{curl} \partial_t \beta^\text{pl}$
  - $= \text{curl}(-\nu \times \alpha)$: evolution equation – generally not closed
  - ... - closed only for special case

Already Kröner was well aware of ...
- the limitations of the dislocation density tensor
- the gap between dislocation physics and continuum plasticity in general

Limitations of the *averaged* density tensor $\alpha = \langle \alpha_i \rangle$
KRÖNER-NYE dislocation density tensor $\alpha$:

\[
\alpha = \text{curl } \beta^{pl}
\]

inhomogeneous plastic distortion causes a dislocation density

\[
\text{div } \alpha = 0
\]
dislocation lines do not start or end inside the crystal

\[
\partial_t \alpha = \text{curl } \partial_t \beta^{pl}
\]
evolution equation – generally not closed

\[
= \text{curl}(-v \times \alpha)
\]
... - closed only for special case

Limitations of the \textit{averaged} density tensor $\alpha = \langle \alpha \rangle$

In general, $\alpha$ does not fulfill $\partial_t \beta^{pl} = \text{curl}(-v \times \alpha)$ because...

\[
\partial_t \alpha = -\text{curl}(\partial_t \beta^{pl}) = -\text{curl}\left(\sum_c \delta \cdot v \times \xi_c \otimes b\right) = -\text{curl}\left(\sum_c \langle \delta \cdot v \rangle \times \langle \xi_c \rangle \otimes b\right)
\]

Applicability of the \textit{averaged} Kröner-Nye tensor:

- OK: only 1 dislocation present (discrete case)
- OK: smooth line bundles with same tangent vector $\xi$ and velocity $v$
- BUT in general: averaging volume contains lines of different orientation
  \[ \rightarrow \text{averaging yields } \rho_{\text{GND}} < \rho \]
2D Continuum dislocation dynamics (Groma 1997):
continuum theory of straight parallel edge dislocations, e.g. applied to...

Model composite


Dislocation distribution with 2D-DD
Dislocation density with 2D CDD


Plastic strain with Groma CDD

FE-mesh distortion with 2D-DD

Plastic strain with a model by M. Gurtin 2002


Hochrainer’s Continuum Dislocation Dynamic (CDD) theory

(Hochrainer, Zaiser & Gumbsch, Phil. Mag. 87 (2007))

distinguish line segments according to their line orientation $\varphi$
$\rightarrow$ lift of the line: $c(x, y) \rightarrow C(x, y, \varphi)$

average over the lift of the dislocation line ("controlled averaging")

higher-dimensional continuum field description of dislocation microstructure

continuous representation of dislocation flow
$\rightarrow$ dislocation density tensor of 2nd order $\alpha^{ij}$

Spatial dislocation loop (red) and lifted dislocation loop in the configuration space (blue)

tangent $L = \frac{dc}{ds} = \left( \frac{dc}{ds} \right) k(x)$

Two main ingredients for the 'lift':

1. generalised line direction $L$
   $$L_{(n, \varphi)} = (\cos \varphi, \sin \varphi, k_{(n, \varphi)})$$

2. generalised velocity $V$
   $$V_{(n, \varphi)} = (v \sin \varphi, -v \cos \varphi, \Theta_{(n, \varphi)})$$

with

- $k$: lines’ curvature
- $\Theta$: rotational velocity
- causing a line to move in orientation direction = rotation
- basically a velocity gradient along the line
Can we obtain a closed form of the evolution equation for $\alpha^{II}$?

$$\partial_t \alpha^{II} = -\text{curl} \left( \sum_C \delta cV \times \frac{dc}{ds} \otimes b \right)$$

...holds under much weaker assumptions: Dislocations in an averaging volume with same line direction...

1. must have the same curvature
2. glide with the same magnitude of velocity

KRÖNER: dislocation lines in an averaging volume must have same line direction $\phi$ the same velocity $v$

CDD side-by-side with the classical Kröner theory

<table>
<thead>
<tr>
<th></th>
<th>CDD</th>
<th>Kröner</th>
</tr>
</thead>
<tbody>
<tr>
<td>formal density</td>
<td>$\delta_c(r) = \int_c \delta(c(s)-r) , ds$</td>
<td>$\delta_c(r) = \int_c \delta(C(s)-(r, \phi)) , ds$</td>
</tr>
<tr>
<td>dislocation density</td>
<td>$\alpha_{c(r)} = \left( \sum_c \delta_c \frac{dC}{ds} \otimes b \right)$</td>
<td>$\alpha^\Pi_{(r, \phi)} = \left( \sum_c \delta_c \frac{dC}{ds} \otimes b \right)$</td>
</tr>
<tr>
<td>scalar dislocation</td>
<td>$\rho = \left| \sum_c \delta_c \frac{dC}{ds} \right|$</td>
<td>$\rho = \left| \sum_c \delta_c \frac{dC}{ds} \right|$</td>
</tr>
<tr>
<td>(average spatial/generaized)</td>
<td>$l = \left( \sum_c \delta_c \frac{dC}{ds} \right) \rho^{-1}$</td>
<td>$L = \left( \sum_c \delta_c \frac{dC}{ds} \right) \rho^{-1} = (l, k)$</td>
</tr>
<tr>
<td>line direction</td>
<td>$\alpha_{c(r)} = \rho l \otimes b$</td>
<td>$\alpha_{(r, \phi)} = \rho L \otimes b$</td>
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<td>discrete disloc. current</td>
<td>$J^d = \sum_c \delta_c \epsilon^d \frac{dC}{ds} \otimes b$</td>
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<tr>
<td>dislocation density</td>
<td>$\alpha_{I(\phi)} = \sum_c \delta_c \frac{dC}{ds} \otimes b$</td>
</tr>
<tr>
<td>Evolution of disloc. density</td>
<td>$\partial_t \alpha_{I(\phi)} = -\text{curl} \left( J^d \right)_{\delta_c}$</td>
</tr>
</tbody>
</table>

→ both theories are very similar from a formal point of view

What do the governing equations look like?

→ if only dislocation glide in a slip plane is considered, $\alpha^{II}$ can be expressed in terms of scalar density $\rho$, $L(r, \phi)$, and average curvature $k$:

$$
\alpha^{II}(r, \phi) = \rho(r, \phi) L(r, \phi) \otimes b = \left[ \rho(r, \phi) \cdot l^x(\phi) \right] \otimes b
= \left[ \rho(r, \phi) \cdot l^y(\phi) \right] \otimes b
= \left[ \rho(r, \phi) \cdot k(r, \phi) \right]
$$

with $r \in \mathbb{R}^2$, $\phi \in [0..2\pi)$

\begin{align*}
l^x(\phi) &= \cos \phi \\
l^y(\phi) &= \sin \phi
\end{align*}

components of tangent to the spatial loop

$\rho$: scalar density

$k$: average curvature

$L$: higher-dimensional line direction

$b$: Burgers vector
evolution of dislocation density tensor $\text{2}^{\text{nd}}$ order $\alpha^{II}$ can be substituted by two scalar evolution equations:

1) evolution of scalar density $\rho$

$$\partial_t \rho = -\text{div} (\rho v) - \partial_x (\rho \vartheta) + \rho v k$$

2) evolution of mean dislocation curvature $k$

$$\partial_t k = -v k^2 + \nabla_L (\vartheta) - \nabla_V k$$

- $\rho$ and $k$ both live in the configuration space $\mathbb{R} \times \mathbb{R} \times S$
- $v$ must be given, e.g. $v = \frac{b}{B} \left( \tau_{\text{m}} - \tau_b \pm \tau_f \right)$ if $|\tau_{\text{m}} - \tau_b| \geq \tau_f$

→ closure problem of dynamics

1) Sandfeld et al., Phil. Mag. (90) 2010
Exploring the 2D system: interpreting the components of the evolution equations

evolution of curvature: \( \partial_t k = -vk^3 + \nabla_z (\theta) - \nabla_z (k) \)

- \( \partial_t k = -vk^3 \) ... change of curvature as for an expanding/shrinking circular loop
- \( \partial_t k = \cdots + \nabla_z (\theta) \) ... change of rotational velocity along the lifted line (2nd derivative of velocity)
- \( \partial_t k = \cdots \nabla_z (k) \) ... curvature change in direction of motion (Euler)

Expanding loop with anisotropic velocity, \( v = f(\phi) \)

- \( \phi \): rotational velocity (vertical component of generalized velocity)
- \( -\phi = \nabla_z (\psi) = \cos \psi \partial_z v - \sin \psi \partial_z v + k \partial_z v \)
- for \( v = \text{const} \) (expanding/shrinking loop):
  \( \partial_z (\psi) = 0 \)
- anisotropic velocity \( v = f(\phi) \):
  \( \partial_z (\psi) \neq 0 \! \)
Expanding loop: anisotropic velocity, $v = f(\varphi)$

\[ \dot{x} = -\text{div}(\sigma) \quad \dot{\varphi} = \sigma \theta + \sigma v, \quad \dot{\varphi} k = -vk^2 - \nabla \varphi \cdot \nabla \varphi \]

(spatial) velocity along the line

rotation of a line segment (red) during loop expansion due to anisotropic velocity law

Expansion of a dislocation loop in an anisotropic stress field

Initial density distribution:

Evolved density distribution:

(a) scalar density $\rho(x,y) = \int_0^L \rho(x,y) \, dy$

(b) projected density $\rho(x,\varphi) = \int_0^L \rho(x,y) \, dy$

(c) scalar density $\rho(x,y)$ at $t = 180$

(d) projected density $\rho(x,\varphi)$ at $t = 180$
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Boundary conditions in CDD – analyzing the path of dislocations

in fact, we only have density representing a loop distribution, there are NO discrete dislocation loops!

Boundary conditions in CDD – analyzing the path of dislocations

x-phi plane:

x-y plane:
**Quadratic ‘grain’ with impenetrable boundaries**

**Initial values:**
- circular dislocation loops, radius $r = 2 \mu m$
- all loops stem from e.g. Frank-Reed sources ($\rightarrow$ plastic slip)
- total density $\rho = 0.1 \cdot 10^{13} m^{-2}$

**Velocity and boundaries:**
- $v = \text{const}$ in the inner part
- $v \rightarrow 0$ in the boundary region

**Boundary conditions are realized as flux boundary conditions**

**Dislocation field quantities for the left half of a quadratic cell ($l=20\mu m$) with impenetrable boundaries:**

- total density $\rho_t$ [$10^{13} m^{-2}$]
- GND density $\alpha$ [$10^{13} m^{-2}$]
- curvature $k$ [$\mu m^{-1}$]
- plastic slip $\gamma$ [-]

**Initial values:**
- circular dislocation loops, radius $r = 2 \mu m$
- total density $\rho = 0.1 \cdot 10^{13} m^{-2}$

**Velocity and boundaries:**
- constant velocity in the inner part
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Quadratic 'grain' with impenetrable boundaries

Dislocation field quantities for the left half of a quadratic cell (l=20μm) with impenetrable boundaries:

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Quadratic 'grain' with impenetrable boundaries

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$\rho$, $\bar{k}$, $|\kappa|$, $\gamma$

Blanckenhagen et al., *Acta Mater* 52, 2004

**Simplifying the higher-dimensional CDD**
Reduction of degrees of freedom by integration over orientation space (HOCHRainer et al. 2009, AIP Conf. Proc. 1168(1))

\[
\alpha^\Pi(r, \phi) = \rho(r, \phi) L(r, \phi) \otimes b = \begin{cases} 
\rho(r, \phi) \cdot l^x(\phi) \\
\rho(r, \phi) \cdot l^y(\phi) \\
\rho(r, \phi) \cdot k(r, \phi)
\end{cases} \otimes b 
\] 

with \( r \in \mathbb{R}^2 \) and \( \phi \in [0..2\pi) \)

\[
l^x(\phi) = \cos \phi \\
l^y(\phi) = \sin \phi
\]

\[
\rho^1(r) = \int_0^{2\pi} \rho(r, \phi) \, d\phi, \\
q^1(r) = \int_0^{2\pi} \rho(r, \phi) \cdot k(r, \phi) \, d\phi, \\
\kappa^1(r) = \int_0^{2\pi} l^x(\phi) \rho(r, \phi) \, d\phi, \\
\kappa^2(r) = \int_0^{2\pi} l^y(\phi) \rho(r, \phi) \, d\phi
\] 

\( \in \mathbb{R}^2 \)

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Reduction of degrees of freedom by integration over orientation space (HÖCHRAINER et al. 2009, AIP Conf. Proc. 1168(1))

\[ \alpha^{II}(r, \varphi) = \rho(r, \varphi) L(r, \varphi) \otimes b = \begin{cases} \rho(r, \varphi) \cdot l^x(\varphi) \\ \rho(r, \varphi) \cdot l^y(\varphi) \\ \rho(r, \varphi) \cdot \kappa(r, \varphi) \end{cases} \otimes b \quad \text{with} \quad r \in \mathbb{R}^2, \quad \phi \in [0..2\pi], \quad l^x(\varphi) = \cos \varphi, \quad l^y(\varphi) = \sin \varphi \]

\[ \kappa^1(r) = \int_{0}^{2\pi} l^x(\varphi) \rho(r, \varphi) \, d\varphi, \quad \kappa^2(r) = \int_{0}^{2\pi} l^y(\varphi) \rho(r, \varphi) \, d\varphi \]

11 and 12-components of Kröner tensor \( \alpha \)

Derivation of simplified evolution equations:

\[ \partial_t \alpha^{II}(r, \varphi) = -\operatorname{curl}(V(r, \varphi) \times \alpha^{II}(r, \varphi)) \]

Assumption: \( v \) and \( k \) are orientation independent

\[ \partial_t \rho^t = -\left( \partial_x (v k^2) - \partial_y (v k^1) \right) + v \rho^t \kappa \]
\[ = -\text{div}(v \kappa^t) + v \rho^t \kappa \]

\[ \partial_t \kappa = \left( -\partial_y (v \rho^t), -\partial_x (v \rho^t) \right) \]

\[ \partial_t \kappa = -v \kappa^2 - \frac{1}{2} \left( \rho^t + \rho^G \right) \nabla_k v + \frac{\rho^t - \rho^G}{\rho^t} \nabla_{\rho^t} v - \frac{1}{\rho^t} (k \nabla_{k^t} v - v \nabla_{k^t} \kappa) \]
dislocation velocity \( v = \frac{b}{B} \tau_{\text{res.}} \) with \( b \): Burgers vector, \( B \): drag coefficient

\[
\tau_{\text{resulting}} = \begin{cases} 
\text{sgn}(\tau_0)\left(|\tau_0| - |\tau_y|\right) & \text{if } |\tau_0| > |\tau_y| \\
0 & \text{otherwise}
\end{cases}
\]

where \( \tau_0 = \tau_{\text{ext}} + \tau_{\text{sc}} - \tau_b - \tau_{\text{lt}} \)

\( \tau_{\text{ext}} \): resolved shear stress (from elastic BVP, e.g. strain rate)
\( \tau_{\text{sc}} \): self-consistent stress (from elastic eigenstrain problem)

\( \tau_b \approx \nabla \rho^6 \): back stress (short range interaction) (proportional to gradient of GND density)

\( \tau_b = Gb^2k \): line tension (short range interaction) (proportional to average curvature)

\( \tau_y \approx \rho^{1/2} \): yield stress (proportional to square root of total density)

**Homogeneous loop distribution with Taylor-type hardening**

\[
\frac{\partial k}{\partial r} = \rho_0 k, \quad \frac{\partial k}{\partial \gamma} = \frac{2\pi}{\rho_0 k} \int \mu v d\varphi, \quad \frac{\partial \gamma}{\partial t} = \int (0.5\mu b) \sqrt{\rho(t)} d\tau(t)
\]

where \( \int \cdot \cdot \cdot d\varphi = \begin{cases} \cdot \cdot \cdot & \text{for } \cdot > 0 \\
0 & \text{otherwise.}
\end{cases} \)

- homogeneous distribution of circular loops, radius \( r \)
- quasi-static loading conditions
- homogeneous distribution of parallel glide planes
- different initial dislocation radii / const. initial density \( \rho_0 = 5 \cdot 10^{11} \text{ m}^{-2} \)
- \( \mu = 10 \text{ GPa}, B = 50 \text{ Pa s}, b = 0.22 \text{ nm} \)
Bending of a thin, single crystalline film

**Figure 1: Geometry**

- • "flux" boundary conditions similar to DDD
- • 2 slip systems
- • 2 microscopic stress components:
  - Taylor-type yield stress and back stress
- • quasi-static load steps

**Figure 2: size effect and strain profiles**

- • only 2 free parameter for Taylor-type yield stress and back stress (quality: almost universal constants)
- • strain profiles and size-effect reproduced (match with experiments and DDD)

Mechanical annealing (‘dislocation starvation’) :

- compression test of a nano pillar
- 2 symm. slip systems
- initially homogenous loop distribution
- open boundaries
- Taylor-type flow stress

- initially: elastic response (regime I)
- plastic activity = loop expansion/outflux (regime II)
- all dislocations left = elastic response (regime III)

Total dislocation density

Intermittent plastic behavior:
Dislocation nucleation within the CDD theory
• **Point of departure for derivation:** higher-dimensional CDD

• **Extra source-terms:**
  \[ \partial_t \rho_{src} = \dot{q} k_{src} \quad \text{and} \quad \partial_t k_{src} = \frac{\dot{q}}{\rho} k_{src} (k_{src} - k) \]
  \[ \dot{q} : \text{rate of plastic deformation due to sources} \]
  \[ k_{src} : \text{initial curvature of newly formed loops} \]

Averaging the above CDD evolution equations yields the **simplified CDD equations with sources:**

\[
\begin{align*}
\partial_t \rho^t &= -\text{div}(v\kappa^t) + v\rho^t \kappa + \dot{\rho}_{src} \\
\partial_t \kappa &= (-\partial_y(v\rho^t + \dot{\rho}_{src}/\kappa_{src})), -\partial_x(v\rho^t + \dot{\rho}_{src}/\kappa_{src}) \\
\partial_t \kappa^t &= -v\kappa^2 - \frac{1}{2} \left( \frac{\rho^t + \rho^G}{\rho^t} v_l l + \frac{\rho^t - \rho^G}{\rho^t} v_l l v \right) - \frac{1}{\rho^t} \left( \kappa \nabla_{\kappa^t} v - v \nabla_{\kappa^t} \kappa \right) \\
&\quad \ldots + \dot{\rho}_{src} (k_{src} - \kappa) - \Delta \dot{\rho}_{src} / (2k_{src})
\end{align*}
\]

**Two possible types of nucleation processes in CDD**

1. **Quasi-discrete sources**
   - similar to the discrete Frank-Read sources
   - ...but: initial bowing-out is not considered
   - continuum source emitts density with curvature and plastic slip
     \( \Delta \rho_{src} \) instead of \( \dot{\rho}_{src} \) !
   - activation of the source: *average* shear stress under source area > critical shear stress
Two possible types of nucleation processes in CDD:

2. Continuous “source field"

- average description of many Frank-Read sources
- each point of the continuum: influx of density ($\dot{\rho}_S$) with certain curvature (and plastic slip)

Temporal averaging results in:
- no discrete activation
- (after some threshold...) influx proportional to local stress

\[ \dot{\rho}_S = \rho v \]

- System:
  - 0-dimensional = homogeneous in x and y direction
  - constant strain rate $\dot{\epsilon} = 10^3 s^{-1} \Rightarrow \dot{\epsilon}_{ext} = G\dot{\epsilon}$
  - material parameter: $b=0.256$ nm, $B=5.6GPa/\mu m$, $E=128GPa$, $v=0.33$, $T=4GPa/nm^2$, $a=0.4$

- Velocity is a function of $\tau_{ext}$, $\tau_{lt} = Tk$ and $\tau_y = aGb\sqrt{\rho}$

- Evolution equations ($\dot{\rho}_S$ such that 1 loop is emitted upon activation)
  - source inactive:
    \[ \dot{\rho}_i = \nu_0 \dot{\rho}_i \]
    \[ \dot{k} = -v\dot{k}^2 \]
    \[ \dot{\gamma} = \nu_0 |\gamma| \]
  - source active:
    \[ \dot{\rho}_i = \nu_0 \dot{\rho}_i + \dot{\rho}_s \]
    \[ \dot{k} = -v\dot{k}^2 + \frac{\dot{\rho}_s (\kappa - k)}{\rho_i} \]
    \[ \dot{\gamma} = (\nu_0 + \dot{\rho}_s / k_i) |\gamma| \]

- Initial conditions:
  - $\rho_{i,0} = 10^{12} m^{-2}$ $k_0 = 0.005$ nm ($r_0 = 200$ nm)
  - $\rho_i = 10^{12} m^{-2}$ $k_0 = 0.00455$ nm ($r_0 = 220$ nm)

- Source parameters:
  - $\tau_{out} = \frac{T}{b \cdot l_{sc}} = 58.85$ MPa
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Curvature $k$ and density $\rho$ change over time $t$.

**Resulting**

- $\tau_{ext}$ increases while simultaneously...
- $\tau_{res}$ decreases $\rightarrow$ source gets shut down
- $\tau_{res} > \tau_{ext}$: $\rightarrow$ source firing: $\rho_{tot}$ and $k$ increase
- $\tau_{res} = \tau_{ext} - \tau_{lt} - \tau_{yield}$ changes due to evolving microstructure

**Loops are just a sketch, we use DENSITY!!**

(a) total density $\rho$ and average curvature $k$

(b) stress components: yield stress $\gamma_y$, line tension $\gamma_l$ and resulting stress $\tau_{res}$

(c) evolution of accumulated plastic strain $\gamma$

(d) stress $\gamma_l + \gamma_y$ vs plastic strain
Quasi-discrete sources: the system

- circular slip plane (radius $R$) with constant resolved shear stress $\tau_{\text{ext}}$
- pure edge source, source length $l_{\text{src}}$
- critical stress for activation $\tau_{\text{crit}} = \frac{2Gb}{l_{\text{src}}}$
- line tension (simplified): $\tau_{\text{lt}} = Gb^2k$ with the dislocations’ curvature $k=1/r$
- pure edge source: loop diameter $\sim 1.5 \times l_{\text{src}}$

Here, the back stress plays an important role for activating the source.
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J. Senger 2009, PhD thesis
...some qualitative comparisons

Snapshots of density evolution in a 2D source distribution
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snapshots of accumulated plastic slip in a 2D source distribution

Summary and Outlook

- CDD contains the kinematics of curved lines - line curvature is THE key ingredient!
- Intermittent plastic activity as well as continuous fluxes can be represented in a straightforward manner
- Boundary conditions can be easily included
- hdCDD even allows for anisotropic velocity law (e.g. edge/screw anisotropy or mixed FR source)

- More work to be done on systematically simplifying evo. eqns + benchmarking
- Full FE coupling with full FCC slip systems
- Analyzing DDD configurations to extract internal stresses, short/long-range stress